

$$A = \bigwedge_{z \in FX} X$$

↳ Galois Closure

$$C \xrightarrow{F} S$$

$$\Gamma_F := \{z, X\} \mid F \xrightarrow{z} X$$

$$\text{III}$$

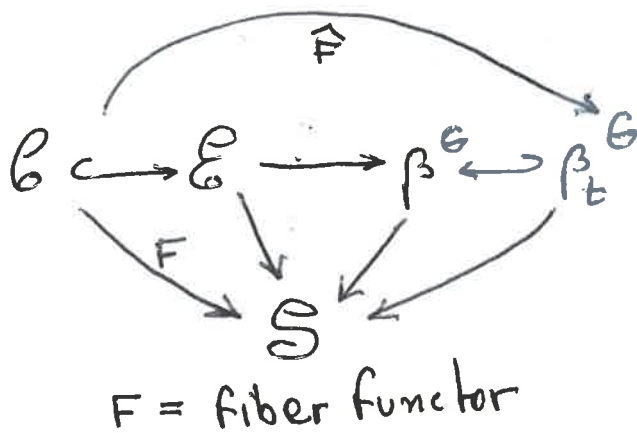
$$z \in FX$$

$$\Gamma_F C (S^C)^{op}$$

Yoneda

GALOIS THEORY

Theorem  $\hat{F}$  is an equivalence



$\mathcal{B}$  site of connected objects

$\mathcal{E}$  atomic topos (connected)

$F$  inverse image of a point  
(recall  $F$  preserves finite limits)

$G$  automorphism group of  $F$

GALOIS-GALOIS

$F$  preserves all limits

$\equiv F$  is representable  $F = [A_0, -]$

$G = \text{Aut}(A_0)^{op}$ ,  $G$  discrete

SGA 1

$F$  finite set valued

$\Rightarrow$  exists Galois closure

$G = \varprojlim_{(a, A)} \text{Aut}(A)^{op}$   $\text{Aut}(A)$  finite  
 $G$  profinite

SGA 4

Galois Topos

$F$  preserves colimits

$\Rightarrow$  exists Galois closure

$G$  is strict progroup  $\{\text{Aut}(A)\}_{(a, A)}^{op}$

$\equiv G = \varprojlim_{(a, A)} \text{Aut}(A)^{op}$  in localic groups  
 $G$  prodiscrete localic

LOCALIC GALOIS

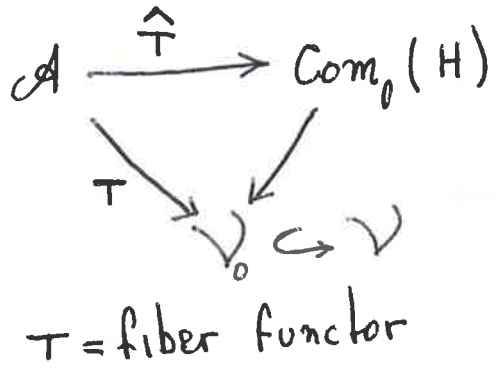
Atomic Topos

$F$  any point

$G =$  localic group of automorphisms

$G$  is a localic group

# TANNAKA THEORY (general)



$V$  cocomplete monoidal closed  
 $V_0$  objects with right dual  
 $H = \text{coalgebra of endomorphisms of } T$

Nat pre-dual (Loyal terminology)

$$\text{Nat}^V(L, T) = \int^X \underbrace{LX \otimes (TX)^\wedge}_{\text{hom}(LX, TX)^\wedge}$$

it follows  $\text{Nat}(L, T) = \text{Nat}^V(L, T)^*$

$$H = \text{End}^V(T) = \text{Nat}^V(T, T)$$

$H^* = V(H, 2) = \text{Nat}(T)$

i)  $H$  is a coalgebra, and there is  $TX \rightarrow \text{End}^V(T) \otimes TX$   
 comodule structure

ii)  $\mathcal{A}$  monoidal,  $T$  monoidal  $\Rightarrow H$  bialgebra

iii)  $\mathcal{A}$  symmetric  $\Rightarrow H$  is commutative algebra

iv) All objects of  $\mathcal{A}$  have duals  $\Rightarrow H$  is a Hopf algebra

Alternative terminology  $\mathcal{A}$  is rigid symmetric tensor

Example  $V = \text{Vec}_K \quad V_0 = V_K^{<\infty}$

Theorem

 $\mathcal{A}$  abelian  $\Rightarrow \hat{T}$  equivalence  
 $T$  faithful

TERMINOLOGY

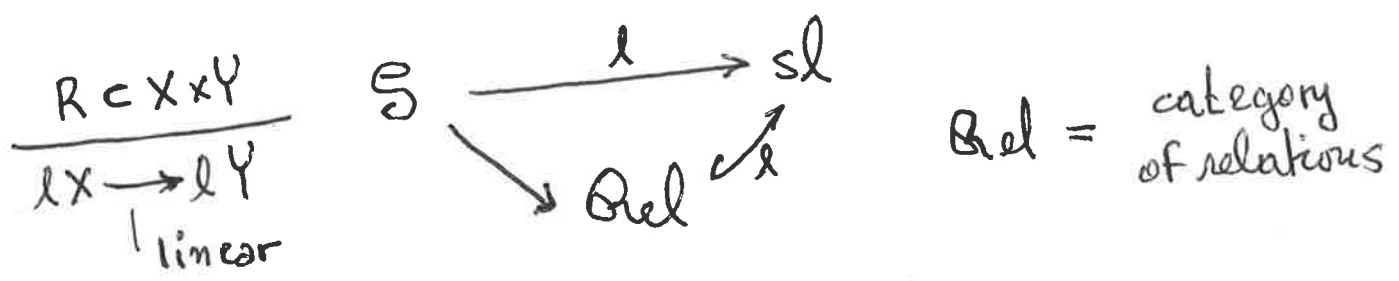
sl monoidal category of sup lattices with linear maps (sup preserving)

Locale . idempotent algebra in sl

Localic group group object in sl<sup>op</sup>

Localic group  $\equiv$  Idempotent Hopf algebra

$X \in \mathcal{S}$   $\mathcal{L}X$  power set with direct image is the free sup lattice on X

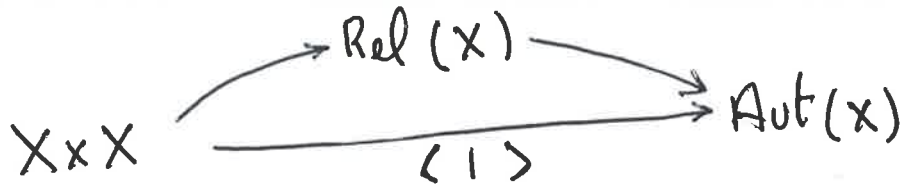


Bijections with values in a locale

- $X \times Y \xrightarrow{\lambda} \Theta$
- ed.  $\bigvee_y \lambda(x, y) = 1$  each  $x$
  - uv.  $\lambda(x, y) \wedge \lambda(x, y') = 0$   $y \neq y'$
  - sv.  $\bigvee_x \lambda(x, y) = 1$  each  $y$
  - in.  $\lambda(x, y) \wedge \lambda(x', y) = 0$   $x \neq x'$

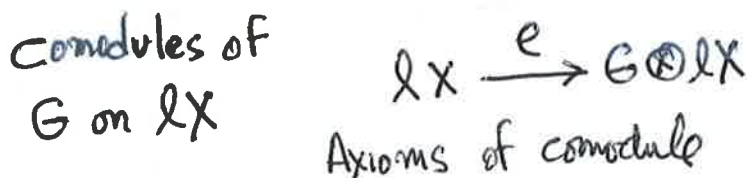
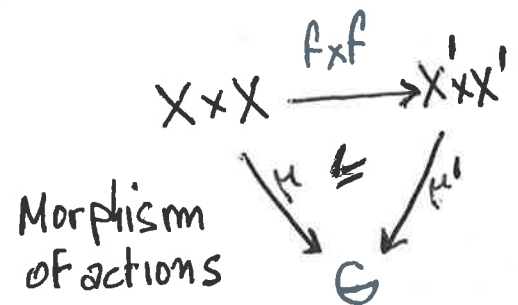
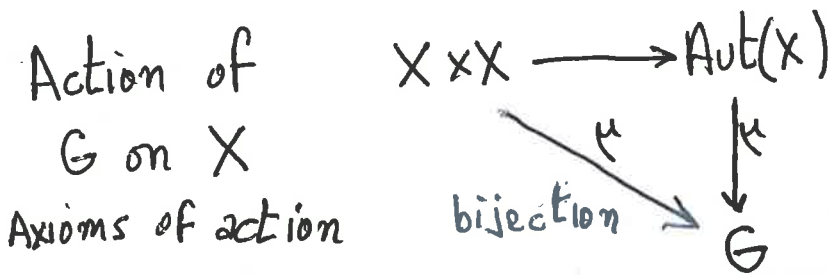
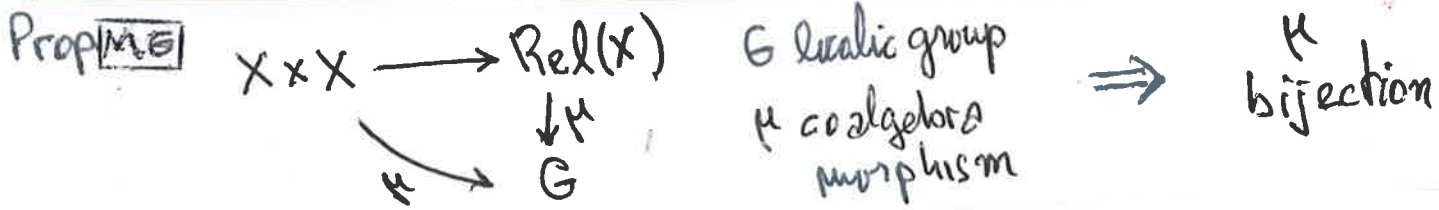
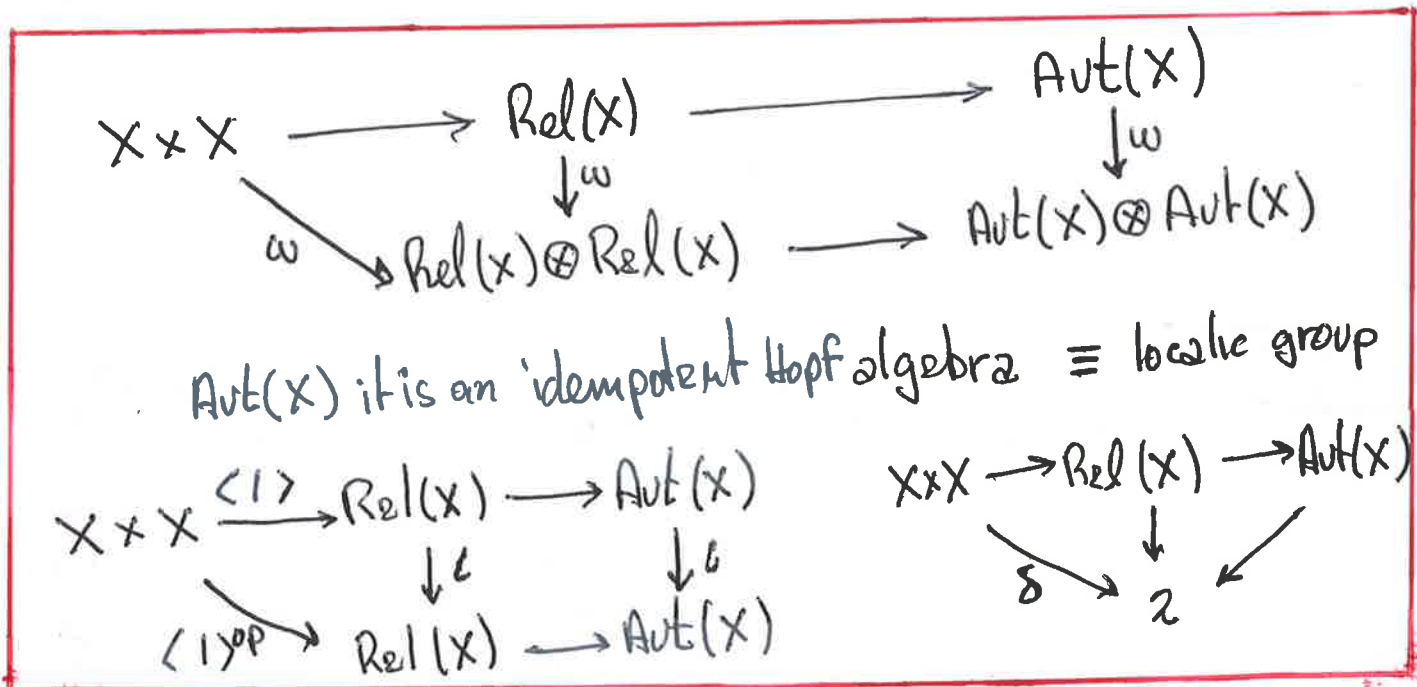
# ACTIONS VERSUS COMODULES

Localic group of automorphisms  $\stackrel{\text{def}}{=} \text{universal bijection}$



$\langle x, y \rangle$  is the subbase of the product topology

Coalgebra structure  $\omega(x, y) = \bigvee_z (x|z) \otimes (z|y)$



Morphism of comodules as usual

$G$  action of  $G$  on  $X \equiv G$  comodule structure on  $\mathcal{L}X$

Proof

$$\begin{array}{l} \text{Rel}(X) \xrightarrow{\mu} G \quad \text{locale morphism} \\ \hline X \times X \xrightarrow{\mu} G \quad \text{relation} \\ \hline \mathcal{L}X \otimes \mathcal{L}X \xrightarrow{\mu} G \quad \text{linear} \\ \hline \mathcal{L}X \xrightarrow{\rho} G \otimes \mathcal{L}X \quad \text{linear} \end{array}$$

Can check  $\rho$   $G$  comodule  $\Leftrightarrow \mu$  coalgebra morphism  
 $\Uparrow \text{Prop } \boxed{MG}$   
 $\mu$   $G$  action

Also Relations between  $G$  sets  $R \twoheadrightarrow X \times Y$  correspond Morphisms of comodules  $\mathcal{L}X \rightarrow \mathcal{L}Y$

Theorem I There is an isomorphism of sl-categories

$$\begin{array}{ccc} \text{Rel}(\beta^G) & \xrightarrow{\cong} & \text{Com}_0(G) \\ & \searrow & \swarrow \\ & \text{Rel} \xrightarrow{\mu} \text{sl}_0 & \end{array}$$

$$\frac{R \twoheadrightarrow X \times Y \text{ in } \beta^G}{\mathcal{L}X \rightarrow \mathcal{L}Y \text{ comodule morphism}}$$

lifts

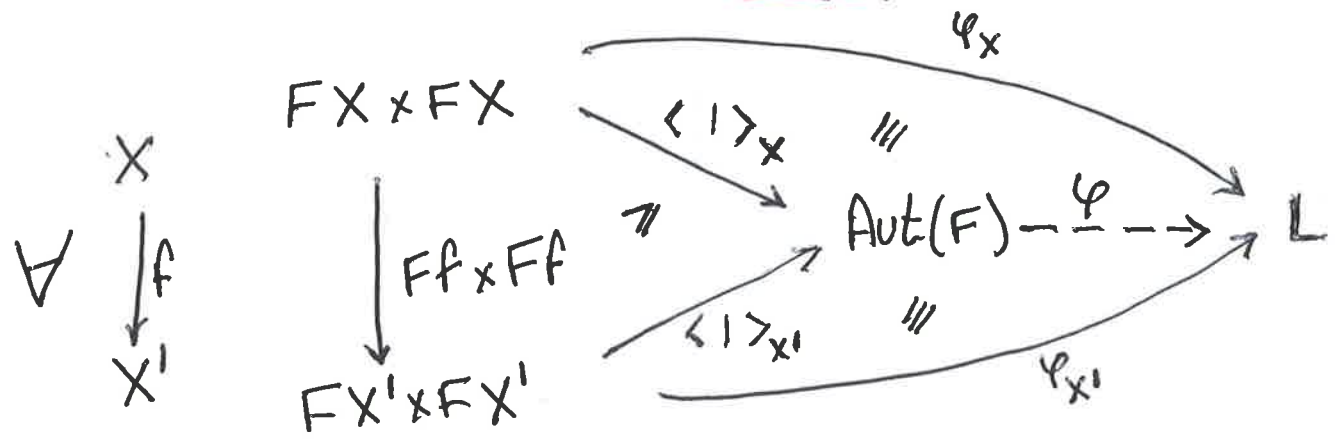
$$\frac{R \twoheadrightarrow X \times Y}{\mathcal{L}X \rightarrow \mathcal{L}Y \text{ linear}}$$

# LOCALIC GALOIS THEORY

## Localic Group of Automorphisms of F

is defined by a universal property in the category of Locales

$$\mathcal{E} \xrightarrow{F} \mathcal{S}$$



▷-cone

$\langle 1 \rangle_x \varphi_x$  bijections,  $\exists! \varphi$  locale morphism.

$\text{Aut}(F) \rightarrow 2$  locale morphism  $\equiv$  Natural isomorphism of F

$$FX \times FX \xrightarrow{\omega_x} \text{Aut}(F) \otimes \text{Aut}(F), \quad FX \times FX \xrightarrow{\langle 1 \rangle_x} \text{Aut}(F), \quad FX \times FX \xrightarrow{\delta_x} 2$$

Defined as before are ▷-cones of bijections

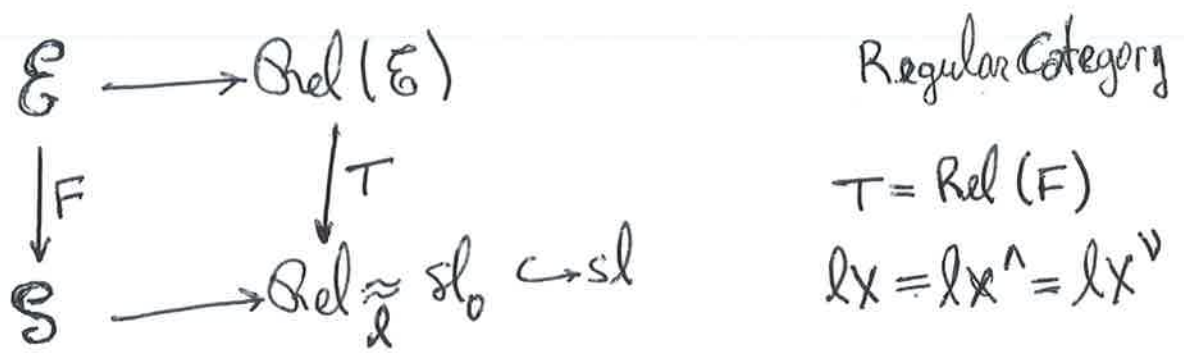
Follows Aut(F) is a localic group and  $\langle 1 \rangle_x$  is an action of  $\text{Aut}(F)$  on  $FX$ , and  $F(F)$  is a morphism of actions.

There is a lifting 
$$\begin{array}{ccc} \hat{F} & \xrightarrow{\beta} & G \\ \mathcal{E} & \xrightarrow{F} & \mathcal{S} \end{array} \quad G = \text{Aut}(F)$$

Theorem  $\mathcal{E} \hookrightarrow \mathcal{E}$  small generators, ▷-cone over  $\mathcal{E}$  suffices  $\Rightarrow \text{Aut}(F)$  exists

Theorem  $\mathcal{E}$  connected atomic,  $\mathcal{E}$  connected generators  $\Rightarrow \hat{F}$  equivalence

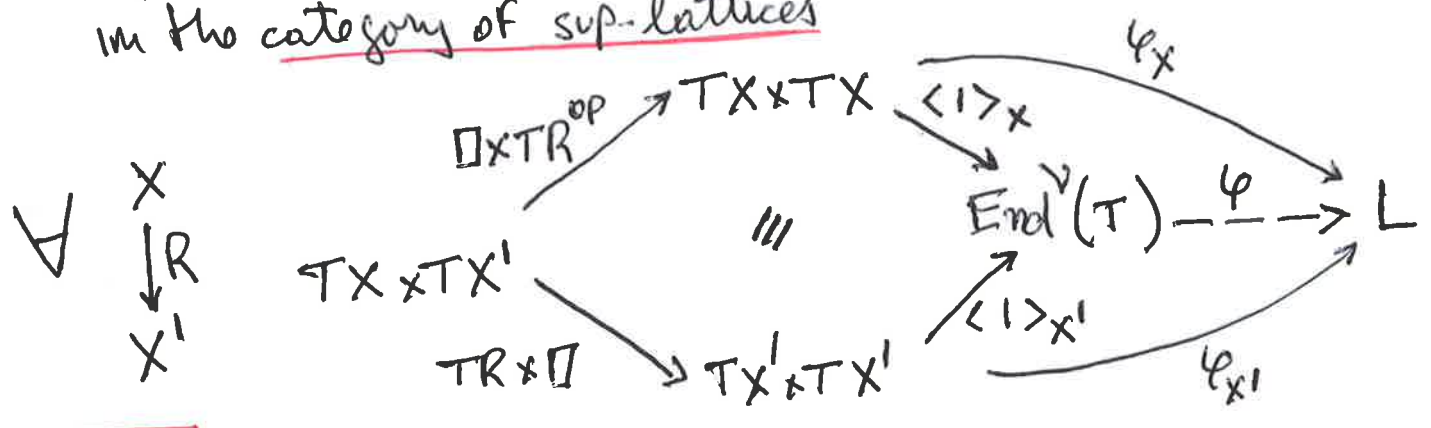
TANAKA CONTEXT FOR GALOIS



Satisfy all conditions in Tannaka General, Thus

$H = \text{End}^\vee(T)$  is a commutative Hopf algebra

By definition  $\text{End}^\vee(T)$  satisfies a universal property in the category of sup-lattices



◇-cone

$\langle 1 \rangle_X, \varphi_X$  relations,  $\exists!$   $\varphi$  linear

By general Tannaka  $\langle x|x' \rangle_X * \langle y|y' \rangle_Y = \langle (x,y)|(x',y') \rangle_{X \times Y}$

Since  $\langle (x,x)|(x',x') \rangle_{X \times X} = \langle x|x' \rangle_X$ ,  $*$  is idempotent

Thus  $\text{End}^\vee(T)$  is a localic group.

Theorem  $\mathcal{B} \xrightarrow{\text{small}} \mathcal{E}$  generators,  $\diamond$ -cone over  $\mathcal{B}$  suffices.  
 $\Rightarrow \text{End}^\vee(T)$  exists



8

The isomorphism  $\text{Aut}(F) \cong \text{End}^V(T)$

Note that  $T=F$  on  $\mathcal{E}$ ,  $TX=FX$   $T(\Gamma_A)=F(f)$ .

Considers families of relations  $TX \times TX \xrightarrow{\lambda_x} L$

Prop  $L$  locale, Any  $\Delta$ -cone of bijections is a  $\diamond$ -cone.

$\Rightarrow \exists! \sigma$  linear  $\text{End}^V(T) \xrightarrow{\sigma} \text{Aut}(F)$

Prop Elements of the form  $\langle x|x' \rangle_x$  generate  $\Rightarrow \sigma$  is a locale morphism.

Prop  $L$  locale, any  $\diamond$ -cone is a  $\Delta$ -cone

The equation  $\lambda_x(xz') \wedge \lambda_y(yz') = \lambda_{x \times y}(xz', yz')$   $\Rightarrow \lambda_x$  are bijections

$\Rightarrow \exists! \rho$  locale morphism  $\text{Aut}(F) \xrightarrow{\rho} \text{End}^V(T)$

Prop It can be checked that  $\sigma$  and  $\rho$  are localic group morphisms

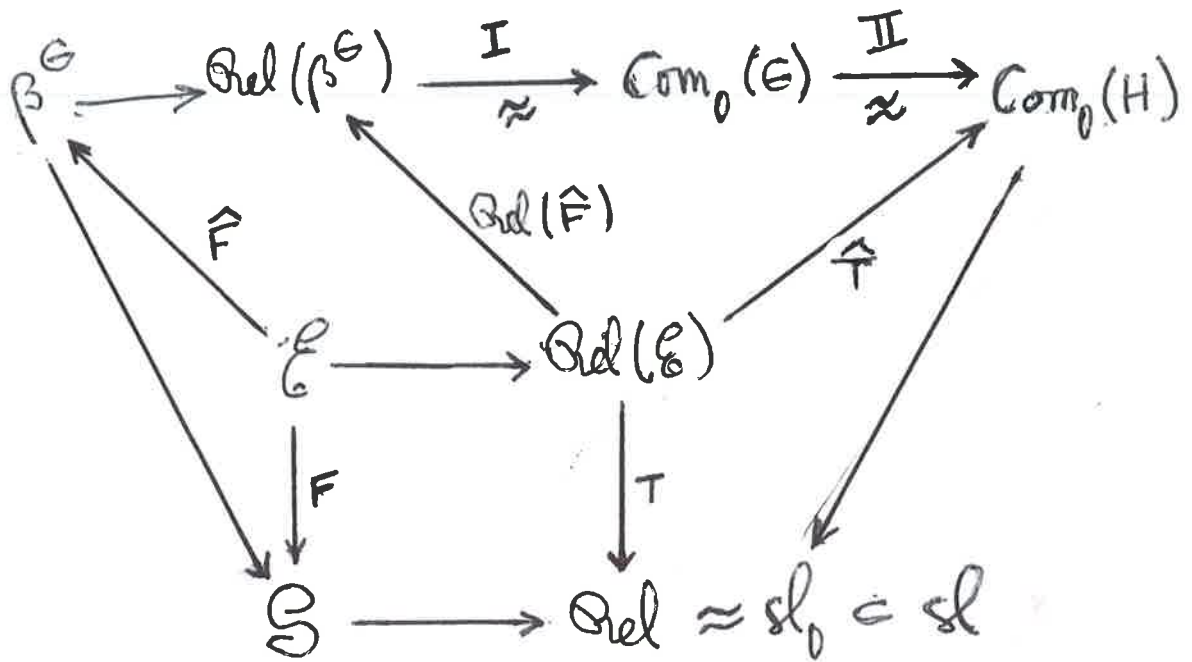
Theorem II

given a pointed regular category

$\mathcal{E} \xrightarrow{F} \mathcal{S}$ ,  $T = \text{Obel}(F)$ ,

there is a unique localic group isomorphism

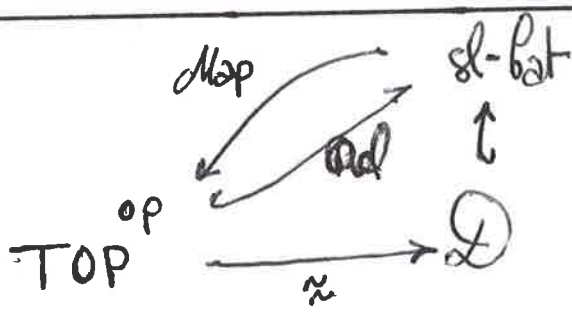
$$\begin{array}{ccc} \text{Aut}(F) & \cong & \text{End}^V(T) \\ \langle 1 \rangle \uparrow & \cong & \uparrow \langle 1 \rangle \\ & & TX \times TX \end{array}$$



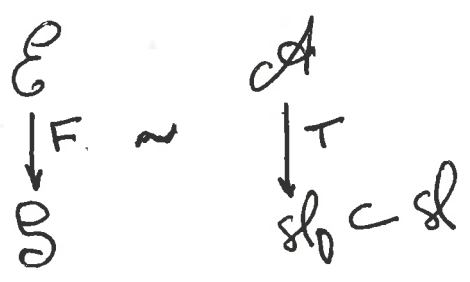
$T = \text{Rel}(F), G = \text{Aut}(F), H = \text{End}^V(T)$

Theorem  $\hat{F}$  equivalence  $\Leftrightarrow \hat{T}$  equivalence

10



$\mathcal{D}$  full image of  $\text{Rel}$   
 Bounded Complete  
 Distributive Category of relations



$\mathcal{E}$  connected  
 $\mathcal{E}$  atomic  
 $F$  open  
 $F$  surjective  
 $\text{Th } A \Updownarrow$   
 $\hat{F}$  equivalence

$\mathcal{A}$  connected  
 $\mathcal{A}$  atomic  
 $T$  open  
 $T$  faithful  
 $\text{Th } B \Updownarrow$   
 $\hat{T}$  equivalence

$\text{Th } A$  is equivalent to  $\text{Th } B$

Since  $\text{Th } A$  holds we have  $\text{Th } B$ . Tannakian Recognition

Como 2018 REFERENCES (ordered by pages)

===== [1]

SGA1 (1960-61), Springer Lecture Notes in Mathematics 224.

SGA4 (1963-64), Springer Lecture Notes in Mathematics 269.

Dubuc E.J., De La Vega C.S., On the Galois theory of Grothendieck, Bol. Acad. Nac. Cienc. Cordoba 65 (2000).

Dubuc E. J., On the representation theory of Galois and Atomic Topoi Journal of Pure and Applied Algebra 186 (2004).

Dubuc E. J., Localic Galois Theory, Advances in Mathematics 175 (2003).

===== [2]

Joyal A., Street R., An Introduction to Tannaka Duality and Quantum Groups, Category Theory, Proceedings, Como 1990, Springer Lecture Notes in Mathematics 1488 (1991).

Szyld M., On Tannaka duality (Tesis de Licenciatura, Dpto. de Mat., F.C.E. y N., Universidad de Buenos Aires (2009)), arXiv:1110.5293v1 [math.CT] (2011).

===== [3] [4] [5] [7] [8] [9]

Dubuc E.J., Szyld M., A Tannakian Context for Galois Theory, Advances in Mathematics 234 (2013), p. 528-549.

===== [6]

Dubuc E.J., Localic Galois Theory, Advances in Mathematics 175 (2003), .

===== [10]

Pitts A., Applications of Sup-Lattice Enriched Category Theory to Sheaf Theory, Proc. London Math. Soc. (3) 37 (1988), 433-480.

Carboni A., Walters F.C., 'Cartesian bicategories, J. Pure Appl. Algebra 49 (1987) 11-32