TOPOSES IN COMO 2018

Eduardo J. Dubuc

$$\mathcal{C} \xrightarrow{F} \mathcal{S}$$

$$\Gamma_F : (x, X) \mid F \xrightarrow{x} X$$

$$\begin{array}{c} ||| \\ x \in FX \end{array}$$

$$\begin{tikzcd}
 & F \arrow[d, "ss"] \arrow[ddl, "a"'] \arrow[ddddl, bend right, "x"'] \arrow[ddd, "z"] \arrow[ddddr, bend left, "y"] & \\
 & A_0 & \\
A \arrow[dd, "\pi_x"] & & \\
 & Z \arrow[dl] \arrow[dr] & \\
X & & Y
\end{tikzcd}$$

$$\Gamma_F \subset \left(\mathcal{S}^{\mathcal{C}}\right)^{op}$$

Yoneda

$$A = \bigwedge_{x \in FX} X$$

Galois Closure

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GALOIS THEORY

$\boxed{\text{Theorem } \ \hat{F} \text{ is an equivalence}}$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[rrr, bend left, "\hat{F}"] \arrow[dr, "F"'] & \mathcal{E} \arrow[r] \arrow[d] & \beta^{G} \arrow[r, leftrightarrow] \arrow[dl] & \beta_t^{G} \arrow[dll] \\
& \mathcal{S} & &
\end{tikzcd}
$$

F = fiber functor

$\mathcal{C}$ site of connected objects

$\mathcal{E}$ atomic topos (connected)

F inverse image of a point

(recall F preserves finite limits)

G automorphism group of F

GALOIS-GALOIS

F preserves all limits

$\equiv$ F is representable $F = [A_0, -]$

$G = Aut(A_0)^{op}$, G discrete.

SGA 1

F finite set valued

$\Rightarrow$ ∃ists Galois closure

$G = \varprojlim_{(a,A)} Aut(A)^{op}$ Aut(A) FINITE

G profinite.

SGA 4

Galois Topos

F preserves cotensors

$\Rightarrow$ ∃ists Galois closure

G is strict progroup $\{Aut(A)^{op}\}_{(a,A)}$

$\equiv$ $G = \varprojlim_{(a,A)} Aut(A)^{op}$ in localic groups

G prodiscrete localic

LOCALIC GALOIS

Atomic Topos

F any point

G = localic group of automorphisms

G is a localic group

TANNAKA THEORY (general)

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\hat{T}"] \arrow[dr, "T"'] & & \mathrm{Com}_0(H) \arrow[dl] \\ & \mathcal{V}_0 \hookrightarrow \mathcal{V} & \end{tikzcd}$$

T = fiber functor

$\mathcal{V}$ cocomplete monoidal closed

$\mathcal{V}_0$ objects with right dual

H = coalgebra of endomorphisms of T

Nat predual (Joyal terminology)

$$\mathrm{Nat}^{\vee}(L,T) = \int^{X} \underbrace{LX \otimes (TX)^{\wedge}}_{\hom(LX,TX)^{\wedge}}$$

it follows $\mathrm{Nat}(L,T) = \mathrm{Nat}^{\vee}(L,T)^{*}$

$$\boxed{H^{*} = \mathcal{V}(H,\mathbb{1}) = \mathrm{Nat}(T)}$$

$$H = \mathrm{End}^{\vee}(T) = \mathrm{Nat}^{\vee}(T,T)$$

i) H is a coalgebra, and there is $TX \longrightarrow \mathrm{End}^{\vee}(T) \otimes TX$ comodule structure

ii) $\mathcal{A}$ monoidal, T monoidal $\Rightarrow$ H bialgebra

iii) $\mathcal{A}$ symmetric $\Rightarrow$ H is commutative algebra

iv) All objects of $\mathcal{A}$ have duals $\Rightarrow$ H is a Hopf algebra

Alternative Terminology $\mathcal{A}$ is rigid symmetric tensor

Example $\mathcal{V} = \mathrm{Vec}_K$ $\mathcal{V}_0 = \mathcal{V}_K^{<\infty}$

$\boxed{\text{Theorem}}$ $\mathcal{A}$ abelian, T faithful $\Rightarrow$ $\hat{T}$ equivalence

TERMINOLOGY

sl monoidal category of sup lattices with linear maps (sup preserving)

Locale. idempotent algebra in sl

Localic group group object in sl^{op}

Localic group $\equiv$ Idempotent Hopf algebra

$X \in \mathcal{S}$ **ℓX** power set with direct image is the free sup lattice on X

$$\frac{R \subset X \times Y}{\ell X \xrightarrow[\text{linear}]{} \ell Y}$$

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\ell"] \arrow[dr] & sl \\ & \mathcal{R}el \arrow[u, "\ell"'] \end{tikzcd}$$

$\mathcal{R}el$ = category of relations

Bijections with values in a locale

$X \times Y \xrightarrow{\lambda} G$

ed. $\bigvee_y \lambda(x,y) = 1$ each x

uv. $\lambda(x,y) \wedge \lambda(x,y') = 0$ $y \neq y'$

su. $\bigvee_x \lambda(x y) = 1$ each y

in. $\lambda(x,y) \wedge \lambda(x' y) = 0$ $x \neq x'$

ACTIONS VERSUS COMODULES

Localic group of automorphisms $=_{def}$ universal bijection

$$\begin{tikzcd} & \mathrm{Rel}(X) \arrow[dr] & \\ X\times X \arrow[ur] \arrow[rr, "\langle 1 \rangle"'] & & \mathrm{Aut}(X) \end{tikzcd}$$

$\langle x, y\rangle$ is the subbase of the product topology

Coalgebra structure $\quad \omega(x,y) = \bigvee_z \langle x|z\rangle \otimes \langle z|y\rangle$

$$\begin{tikzcd} X\times X \arrow[r] \arrow[dr, "\omega"'] & \mathrm{Rel}(X) \arrow[r] \arrow[d, "\omega"] & \mathrm{Aut}(X) \arrow[d, "\omega"] \\ & \mathrm{Rel}(X)\otimes \mathrm{Rel}(X) \arrow[r] & \mathrm{Aut}(X)\otimes \mathrm{Aut}(X) \end{tikzcd}$$

Aut(X) it is an idempotent Hopf algebra $\equiv$ localic group

$$\begin{tikzcd} X\times X \arrow[r, "\langle 1 \rangle"] \arrow[dr, "\langle 1 \rangle^{op}"'] & \mathrm{Rel}(X) \arrow[r] \arrow[d, "\iota"] & \mathrm{Aut}(X) \arrow[d, "\iota"] \\ & \mathrm{Rel}(X) \arrow[r] & \mathrm{Aut}(X) \end{tikzcd} \qquad \begin{tikzcd} X\times X \arrow[r] \arrow[dr, "\delta"'] & \mathrm{Rel}(X) \arrow[r] \arrow[d] & \mathrm{Aut}(X) \arrow[dl] \\ & 2 & \end{tikzcd}$$

Prop [M.G]

$$\begin{tikzcd} X\times X \arrow[r] \arrow[dr, "\mu"'] & \mathrm{Rel}(X) \arrow[d, "\mu"] \\ & G \end{tikzcd}$$

G localic group, μ coalgebra morphism $\Longrightarrow$ μ bijection

Action of G on X

Axioms of action

$$\begin{tikzcd} X\times X \arrow[r] \arrow[dr, "\mu"', "\text{bijection}"] & \mathrm{Aut}(X) \arrow[d, "\mu"] \\ & G \end{tikzcd}$$

Morphism of actions

$$\begin{tikzcd} X\times X \arrow[rr, "f\times f"] \arrow[dr, "\mu"'] & & X'\times X' \arrow[dl, "\mu'"] \\ & G & \end{tikzcd}$$

Comodules of G on ℓX

$$\ell X \xrightarrow{e} G \otimes \ell X$$

Axioms of comodule

Morphism of comodules as usual

G action of G on X $\equiv$ G comodule structure on ℓX

proof

$$\frac{\frac{\frac{\mathrm{Rel}(X) \xrightarrow{\mu} G \quad \text{locale morphism}}{X \times X \xrightarrow{\mu} G \quad \text{relation}}}{\ell X \otimes \ell X \xrightarrow{\mu} G \quad \text{linear}}}{\ell X \xrightarrow{\rho} G \otimes \ell X \quad \text{linear}}$$

Can check ρ G comodule $\Longleftrightarrow$ μ coalgebra morphism

$\Updownarrow$ Prop $\boxed{MG}$

μ G action ∎

Also Relations between G sets correspond Morphisms of comodules

$R \rightarrowtail X \times Y$ $\qquad$ $\ell X \longrightarrow \ell Y$

Theorem I There is an isomorphism of sl-categories

$$\begin{tikzcd}
\mathrm{Rel}(\beta^G) \arrow[rr, "\approx"] \arrow[dr] & & \mathrm{Com}_0(G) \arrow[dl] \\
& \mathrm{Rel} \xrightarrow[\ell]{\approx} sl_0 &
\end{tikzcd}$$

$$\boxed{\frac{R \rightarrowtail X \times Y \text{ in } \beta^G}{\ell X \to \ell Y \text{ comodule morphism}}}$$

lifts

$$\boxed{\frac{R \rightarrowtail X \times Y}{\ell X \to \ell Y \text{ linear}}}$$

LOCALIC GALOIS THEORY

$\mathcal{E} \xrightarrow{F} \mathcal{S}$ — **Localic Group of Automorphisms of F** is defined by a universal property in the category of Locales

$$\begin{tikzcd}
X \arrow[d, "f"] & FX \times FX \arrow[d, "Ff \times Ff"] \arrow[dr, "\langle\,\rangle_X"] \arrow[drr, bend left, "\varphi_X"] & & \\
X' & FX' \times FX' \arrow[r, "\langle\,\rangle_{X'}"'] \arrow[rr, bend right, "\varphi_{X'}"'] & \mathrm{Aut}(F) \arrow[r, dashed, "\varphi"] & L
\end{tikzcd}$$

$\forall$

$\boxed{\triangleright\text{-cone}}$

$\langle\,\rangle_X$ φ_X bijections, $\exists!\,\varphi$ locale morphism.

$\mathrm{Aut}(F) \longrightarrow 2$ locale morphism $\equiv$ Natural isomorphism of F

$FX \times FX \xrightarrow{\omega_X} \mathrm{Aut}(F) \otimes \mathrm{Aut}(F)$, $FX \times FX \xrightarrow{\iota_X} \mathrm{Aut}(F)$, $FX \times FX \xrightarrow{\delta_X} 2$

Defined as before are $\triangleright$-cones of bijections

Follows Aut(F) is a localic group and $\langle\,\rangle_X$ is an action of Aut(F) on FX, and F(f) is a morphism of actions.

There is a lifting

$$\begin{tikzcd}
 & \beta^G \arrow[d] \\
\mathcal{E} \arrow[ur, "\hat{F}"] \arrow[r, "F"'] & \mathcal{S}
\end{tikzcd}$$

$G = \mathrm{Aut}(F)$.

Theorem $\mathcal{C} \hookrightarrow \mathcal{E}$ small generators, $\triangleright$-cone over $\mathcal{C}$ suffices $\Rightarrow$ Aut(F) exists

Theorem $\mathcal{E}$ connected atomic, $\mathcal{C}$ connected generators $\Rightarrow$ $\hat{F}$ equivalence

TANAKA CONTEXT FOR GALOIS

$$
\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "F"] & \mathrm{Rel}(\mathcal{E}) \arrow[d, "T"] \\
\mathcal{S} \arrow[r] & \mathrm{Rel} \underset{\ell}{\approx} s\ell_0 \hookrightarrow s\ell
\end{tikzcd}
$$

Regular Category

$T = \mathrm{Rel}(F)$

$\ell X = \ell X^{\wedge} = \ell X^{\vee}$

Satisfy all conditions in Tannaka General, Thus

$H = \mathrm{End}^{\vee}(T)$ is a commutative Hopf algebra

By definition $\mathrm{End}^{\vee}(T)$ satisfies a universal property in the category of sup-lattices

$$
\forall \quad
\begin{tikzcd}
X \arrow[d, "R"] \\
X'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TX \times TX \arrow[dr, "\langle 1 \rangle_X"] \arrow[drr, bend left, "\varphi_X"] & & \\
TX \times TX' \arrow[ur, "\square \times TR^{op}"] \arrow[dr, "TR \times \square"'] & \equiv & \mathrm{End}^{\vee}(T) \arrow[r, dashed, "\varphi"] & L \\
 & TX' \times TX' \arrow[ur, "\langle 1 \rangle_{X'}"'] \arrow[urr, bend right, "\varphi_{X'}"'] & &
\end{tikzcd}
$$

$\boxed{\diamond\text{-cone}}$

$\langle 1 \rangle_X$, φ_X relations, $\exists !_0\ \varphi$ linear

By general Tannaka $\langle x|x' \rangle_X * \langle y|y' \rangle_Y = \langle (x\,y)|(x'\,y') \rangle_{X \times Y}$

Since $\langle (x\,x)|(x'\,x') \rangle_{X \times X} = \langle x|x' \rangle_X$, $*$ is idempotent

Thus $\mathrm{End}^{\vee}(T)$ is a localic group.

Theorem $\mathcal{C} \overset{\text{small}}{\hookrightarrow} \mathcal{E}$ generators, $\diamond$-cone over $\mathcal{C}$ suffices.

$\Rightarrow \mathrm{End}^{\vee}(T)$ exists

The isomorphism $\mathrm{Aut}(F) \approx \mathrm{End}^{\vee}(T)$

Note that $T = F$ on $\mathcal{E}$, $\ {}^{\cdot}TX = {}^{\cdot}FX \quad T(\Gamma_f) = F(f)$.

Consider families of relations $TX \times TX \xrightarrow{\lambda_X} L$

Prop L locale, Any $\triangleright$-cone of bijections is a $\Diamond$-cone.

$\Rightarrow \exists!\ \sigma$ linear $\mathrm{End}^{\vee}(T) \xrightarrow{\sigma} \mathrm{Aut}(F)$

Prop Elements of the form $\langle x | x' \rangle_X$ generate $\Rightarrow$ σ is a locale morphism.

Prop L locale, any $\Diamond$-cone is a $\triangleright$-cone

The equation $\lambda_X(xx') \wedge \lambda_Y(yy') = \lambda_{X \times Y}((xy)(x'y'))$ $\Rightarrow$ λ_X are bijections

$\Rightarrow \exists!$ locale morphism $\mathrm{Aut}(F) \xrightarrow{\rho} \mathrm{End}^{\vee}(T)$

Prop It can be checked that σ and ρ are localic group morphisms

Theorem II

given a pointed regular category $\mathcal{E} \xrightarrow{F} \mathfrak{S}$, $T = \mathcal{R}el(F)$, there is a unique localic group isomorphism

$$\begin{tikzcd}
\mathrm{Aut}(F) \arrow[rr, "\approx"] & & \mathrm{End}^{\vee}(T) \\
& TX \times TX \arrow[ul, "\langle 1 \rangle"] \arrow[ur, "\langle 1 \rangle"'] \arrow[u, phantom, "\parallel\!\!|"] &
\end{tikzcd}$$

$$
\begin{tikzcd}
\beta^G \arrow[r] \arrow[ddr] & \mathrm{Rel}(\beta^G) \arrow[r, "\mathrm{I}", "\approx"'] & \mathrm{Com}_0(G) \arrow[r, "\mathrm{II}", "\approx"'] & \mathrm{Com}_0(H) \arrow[ddl] \\
& \mathcal{E} \arrow[ul, "\hat{F}"] \arrow[r] \arrow[d, "F"] & \mathrm{Rel}(\mathcal{E}) \arrow[ul, "\mathrm{Rel}(\hat{F})"'] \arrow[ur, "\hat{T}"'] \arrow[d, "T"] & \\
& \mathcal{S} \arrow[r] & \mathrm{Rel} \approx sl_0 \subset sl &
\end{tikzcd}
$$

$T = \mathrm{Rel}(F)$, $G = \mathrm{Aut}(F)$, $H = \mathrm{End}^{\vee}(T)$

<u>Theorem</u> $\boxed{\hat{F} \text{ equivalence} \iff \hat{T} \text{ equivalence}}$

10

$$
\begin{tikzcd}
& sl\text{-}\mathrm{Cat} \arrow[dl, bend right, "\mathrm{Map}"'] \\
\mathrm{TOP}^{op} \arrow[ur, bend right, "\mathrm{Rel}"'] \arrow[r, "\approx"'] & \mathcal{D} \arrow[u, hook]
\end{tikzcd}
$$

$\mathcal{D}$ full image of Rel

Bounded Complete Distributive Category of Relations

$$
\begin{tikzcd}
\mathcal{E} \arrow[d, "F"] & \sim & \mathcal{A} \arrow[d, "T"] \\
\mathcal{S} & & sl_0 \subset sl
\end{tikzcd}
$$

$\mathcal{E}$ connected	$\mathcal{A}$ connected
$\mathcal{E}$ atomic	$\mathcal{A}$ atomic
F open	T open
F surjective	T faithful
Th A $\Updownarrow$	Th B $\Updownarrow$
$\hat{F}$ equivalence	$\hat{T}$ equivalence

Th A is equivalent to Th B

Since Th A holds we have Th B. Tannakian Recognition

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