Some glances at topos theory

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Introduction

The notion of *sheaf* on a topological space emerged during the period around the second world war, in order to provide an efficient tool to handle local problems. It admits a straightforward generalization to the case of *locales*: lattices which mimic the properties of the lattice of open subsets of a space. But the striking generalization has been that of a sheaf on a site, that is, a sheaf on a small category provided with a so-called *Grothendieck topology*. That notion became essential in algebraic geometry, through the consideration of *schemes*. In the late sixties, F.W. Lawvere introduced *elementary toposes*: categories satisfying axiomatically the properties typical of the categories of sheaves of sets. Each topos provides a model of intuitionistic logic. Except for what has just been mentioned in this paragraph, we shall stay on the safe side in this text, avoiding to mention whatever paternity of a notion or a result.

These notes intend to give a quick overview of some relevant aspects of topos theory, without entering the details of the proofs. They assume some reasonable familiarity with category theory.

Chapter 1 introduces the sheaves on a topological space and on a locale and exhibits some structural properties of the corresponding *localic topos*. We provide an application to ring theory.

Chapter 2 begins with the notion of a *Grothendieck topology* on a small category and the corresponding notion of sheaf. It investigates some exactness and structural properties of the corresponding *Grothendieck topos*.

In Chapter 3, we switch to the axiomatic of those categories called *elementary toposes* and review some of their important properties.

Chapter 4 throws some light on the *internal logic* of a topos, which is intuitionistic, and the way to use it in order to prove theorems "elementwise".

Chapter 5 investigates the *morphisms of toposes*, both the *logical* ones and the *geometriccal* ones. The link is made with internal notions of topology and sheaf.

We conclude in Chapter 6 with the notion of the *classifying topos* $\mathcal{E}(\mathbb{T})$ of a theory $\mathbb{T}$: a Grothendieck topos which contains a generic model M of $\mathbb{T}$. Generic in the sense that every model of $\mathbb{T}$ in whatever Grothendieck topos $\mathcal{F}$ can be reconstructed from M via the geometric morphism of toposes between $\mathcal{F}$ and $\mathcal{E}(\mathbb{T})$.

I thank *Olivia Caramello* who faced me with the somehow foolish challenge of introducing an audience to topos theory, up to the notion of a classifying topos, through a series of six one hour talks. This has been the genesis of these notes.

Chapter 1

Localic toposes

1.1 Sheaves on a topological space

In a first calculus course, one studies in particular the set of continuous functions on the reals. The sheaf approach to this same question focuses on considering the continuous functions existing at the neighborhood of a point. For example $\log x$ and $\sqrt{x}$ exist at the neighborhood of each $r > 0$, but not at the neighborhood of $r < 0$. Of course, considering open neighborhoods suffices. So our sheaf of continuous functions on the reals consists in specifying, for each open subset $U \subseteq \mathbb{R}$, the set $\mathcal{C}(U, \mathbb{R})$ of real continuous functions on U. Clearly, when $V \subseteq U$ is a smaller open subset, every $f \in \mathcal{C}(U, \mathbb{R})$ restricts as some $f|_V \in \mathcal{C}(V, \mathbb{R})$. Writing $\mathcal{O}(\mathbb{R})$ for the lattice of open subsets of the reals, we get so a contravariant functor

$$\mathcal{C}(-, \mathbb{R}) \colon \mathcal{O}(\mathbb{R}) \longrightarrow \mathsf{Set}$$

to the category of sets.

Next, given two open subsets U, V and continuous functions $f \colon U \longrightarrow \mathbb{R}$, $g \colon V \longrightarrow \mathbb{R}$ which coincide on $U \cap V$, we can "glue" f and g together to extend them in a continuous function defined on $U \cup V$. And of course, the same process holds when choosing an arbitrary number of $f_i \colon U_i \longrightarrow \mathbb{R}$, not just two. So our sheaf of continuous functions satisfies the property:

> Given open subsets $U = \bigcup_{i \in I} U_i$ and continuous functions $f_i \in \mathcal{C}(U_i, \mathbb{R})$
> If for all indices i, j, one has $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$
> Then there exists a unique $f \in \mathcal{C}(U, \mathbb{R})$ such that for each i, $f|_{U_i} = f_i$.

1 **Definition 1.1** *Consider a topological space X and its lattice $\mathcal{O}(X)$ of open subsets.*
A presheaf *F on X is a contravariant functor $F \colon \mathcal{O}(X) \longrightarrow \mathsf{Set}$.*
When $V \leq U$ in $\mathcal{O}(X)$, let us write

$$F(U) \longrightarrow F(V), \quad a \mapsto a|_v$$

for the action of the functor F on the morphism $V \subseteq U$ of $\mathcal{O}(X)$.
A sheaf *F on X is a presheaf satisfying the axiom*

> *Given $U = \bigcup_{i \in I} U_i$ in $\mathcal{O}(X)$ and $a_i \in F(U_i)$ for each i*
> *If for all indices i, j one has $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$*
> *Then there exists a unique $a \in F(U)$ such that for each i, $a|_{U_i} = a_i$.*

The morphisms of preseheaves or sheaves are the natural transformations between them.

The considerations at the beginning of this section extend at once to yield the following examples.

2 **Example 1.2** *Given a natural number $k \in \mathbb{N}$, the mapping*

$$\mathcal{O}(\mathbb{R}) \longrightarrow \mathsf{Set}, \quad U \mapsto \mathcal{C}^k(U, \mathbb{R})$$

associating with an open subset U the set of k-times differentiable functions from U to $\mathbb{R}$, together with the restriction mappings, is a sheaf on $\mathbb{R}$.

3 **Example 1.3** *Given two topological spaces X and Y, the mapping*

$$\mathcal{O}(X) \longrightarrow \mathsf{Set}, \quad U \mapsto \mathcal{C}(U, Y)$$

associating with an open subset U the set of continuous mappings from U to Y, together with the restriction mappings, is a sheaf on X.

Of course in the first example, when $k \leq l$, the sheaf $\mathcal{C}^l(-, \mathbb{R})$ is a subsheaf of $\mathcal{C}^k(-, \mathbb{R})$. In the second example, we obtain at once a subsheaf of $\mathcal{C}(-, Y)$ when considering:

4 **Example 1.4** *Given a continuous mapping $p\colon Y \longrightarrow X$, the mapping*

$$\mathcal{O}(X) \longrightarrow \mathsf{Set}, \quad U \mapsto \mathcal{S}(U, Y) = \{s\colon U \to Y \mid s \in \mathcal{C}(U, Y),\ p \circ s = \mathsf{id}_U\}$$

associating with an open subset U the set of continuous sections of p on U, together with the restriction mappings, is a sheaf on X.

This last example is somehow "generic" since:

5 **Proposition 1.5** *Let F be a sheaf on the topological space X. There exists a topological space Y and a continuous mapping $p\colon Y \longrightarrow X$ such that F is isomorphic to the sheaf of continuous sections of p.*

Sketch of the proof At the very beginning of this section, we insisted that the sheaf of continuous functions on the reals focuses on the behavior of continuous functions at the neighborhood of each point. Given a real number r, we intend thus to consider all continuous real valued functions defined on a neighborhood of r, identifying two such functions when they coincide on a (smaller) neighborhood of r. What we perform so is the filtered colimit

$$S_r = \mathsf{colim}_{\,U \ni r} \mathcal{C}(U, \mathbb{R}).$$

The colimit is indeed filtered since given $r \in U$ and $r \in V$, we have of course $r \in U \cap V$.

This construction generalizes at once to an arbitrary sheaf F on a topological space X. Given a point $x \in X$, the *stalk* of the sheaf F at the point x is defined as the filtered colimit

$$S_x = \mathsf{colim}_{\,U \ni x} F(U).$$

The set Y is then the disjoint union $\coprod_{x\in X} S_x$ of these stalks and the mapping $p\colon Y \longrightarrow X$ is the projection mapping the whole stalk S_x on the point x. Next for every $U \in \mathcal{O}(X)$ and every element $a \in F(U)$, we have a section of p

$$\sigma_a^U : U \longrightarrow Y \quad x \mapsto [a] \in S_x$$

where $[a]$ indicates the equivalence class of a in the colimit. Let us provide Y with the final topology for all these sections σ_a^U, for all $U \in \mathcal{O}(X)$ and all $a \in F(U)$. The mapping p is continuous for that topology and with the notation of Example 1.5, the mappings

$$F(U) \longrightarrow \mathcal{S}(U, Y), \quad a \mapsto \sigma_a^U$$

constitute an isomorphism of sheaves. □

Proposition 1.5 is the essential ingredient of a more precise theorem.

$\boxed{6}$ **Definition 1.6** *A mapping $p\colon Y \longrightarrow X$ between topological spaces is* étale *when, for every point $y \in Y$, there exist open neighborhoods W of y and V of $p(y)$, such that p induces an homeomorphism between W and V.*

It is immediate to observe that an étale mapping is both continuous and open and that a composite of étale mappings is still étale. Moreover, given a commutative triangle of continuous mappings

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

if f and g are étale, so is h. So the category of étale mappings over X is a full subcategory of Top/X.

$\boxed{7}$ **Theorem 1.7** *The category of sheaves on a topological space X is equivalent to the category of étale mappings over X.*

Sketch of the proof Observing that the mapping p of Proposition 1.5 is étale, Example 1.4 and Proposition 1.5 describe the equivalence at the level of objects. □

A classical example of a non trivial étale mapping is the projection of a circular helix on its base circle. A more trivial example is the codiagonal $\nabla\colon X \coprod X \longrightarrow X$ of the coproduct.

Let us conclude this section with emphasizing how strong is the notion of étale mapping. The inclusion of a subspace provided with the induced topology is generally not an étale mapping ... except when the subspace is open. For example the inclusion of the real line in the real plane is by no means étale.

1.2 Sheaves on a locale

Definition 1.1 shows at once that the notion of sheaf on a topological space depends only on the corresponding lattice of subobjects. So one would be tempted to extend this definition to the case of an arbitrary complete lattice: complete, since the definition of a sheaf on a topological space uses a condition like $U = \bigcup_{i\in I} U_i$. But completeness does not suffice. Indeed the notion of sheaf on a topological space uses also in an essential way the restriction to a smaller open subset $V \subseteq U$. Of course since finite intersections and arbitrary unions of open subsets are computed set theoretically, one gets at once

$$V = V \cap U = V \cap \bigcup_{i\in I} U_i = \bigcup_{i\in I} (V \cap U_i)$$

so that the covering of U restricts as a covering of V. Such a property is essential in many of the proofs above.

We shall thus adopt the following definition:

$\boxed{8}$ **Definition 1.8** *A* locale *is a complete lattice in which finite meets distribute over arbitrary joins.*

The condition in the definition is thus

$$u \wedge \bigvee_{i\in I} v_i = \bigvee_{i\in I} (u \wedge v_i)$$

for all elements of the locale. Notice that a locale has a top element 1 (the join of all its elements) and a bottom element 0 (the join of the empty family of elements). It has also arbitrary meets (the join of all the lower bounds), but this is of little interest since infima in a locale do not have any relevant property; in the case of topological spaces, these infima are the interior of the set theoretical intersection.

Of course we define now:

$\boxed{9}$ **Definition 1.9** *Consider a locale L*
A presheaf *on L is a contravariant functor $F\colon L \longrightarrow \mathsf{Set}$.*
When $v \le u$ in L, let us write

$$F(u) \longrightarrow F(v), \quad a \mapsto a|_v$$

for the action of the functor F on the morphism $v \le u$ of L.
A sheaf *on L is a presheaf F satisfying the axiom*

> *Given $u = \bigvee_{i\in I} u_i$ in L and $a_i \in F(u_i)$ for each i*
> *If for all indices i, j one has $a_i|_{u_i\wedge u_j} = a_j|_{u_i\wedge u_j}$*
> *Then there exists a unique $a \in F(u)$ such that for each i, $a|_{u_i} = a_i$.*

The morphisms of presheaves or sheaves are the natural transformations between them.

The families $(a_i)_{i\in I}$ as in Definition 1.9 are generally referred to as *compatible families of elements* along the covering $u = \bigvee_{i\in I} u_i$; $a \in F(u)$ is called the *gluing* of that family.

Everything has of course been done to get as a first example:

10 **Example 1.10** *Given a topological space X, its lattice $\mathcal{O}(X)$ of open subsets is a locale and the notions of sheaf on the space X and sheaf on the locale $\mathcal{O}(X)$ coincide.* □

10a **Example 1.11** *Each representable functor on a locale is a sheaf on that locale.*

Sketch of the proof If u is an element of a locale L, the corresponding representable functor has value the singleton on each $v \leq u$, and the empty set elsewhere. □

10b **Example 1.12** *Given a locale L,*

$$\Omega(u) = \downarrow u = \{v \in L | v \leq u\}$$

is a sheaf on L.

Sketch of the proof A compatible family $v_i \in \downarrow u_i$ glues as $\bigvee_{i \in I} v_i$ in $\downarrow(\bigvee_{i \in I} u_i)$. □

11 **Example 1.13** *Every complete Boolean algebra is a locale.*

Sketch of the proof A Boolean algebra B is in particular a distributive lattice. It follows easily that given three elements u, v, w in B

$$(u \wedge v) \leq w \quad \mathsf{iff} \quad u \leq (\complement v \vee w).$$

This shows that the functor $- \wedge v \colon B \longrightarrow B$ admits the right adjoint $\complement v \vee -$, thus preserves all colimits. □

The following result will play an essential role when studying the internal logic of toposes.

12 **Theorem 1.14** *Every locale is a Cartesian closed category.*

Sketch of the proof Consider three elements u, v, w in a locale L. Put

$$(v \Rightarrow w) = \bigvee \{x \in L \mid x \wedge v \leq w\}.$$

It follows at once that

$$(u \wedge v) \leq w \quad \mathsf{iff} \quad a \leq (v \Rightarrow w).$$

But $u \wedge v$ is the product of u and v in the category L. We have thus observed that the functor $(- \wedge v)$ admits $(v \Rightarrow -)$ as a right adjoint. □

The notation $v \Rightarrow w$ in Theorem 1.14 calls for a comment. Imagine a moment that the elements of the locale L are in fact statements in a theory that we are developing. Considering the usual logical connectors **and**, **or**, **implies**, …, a well-known rule of mathematical logic is:

From (u **and** v) one can infer w
if and only if
from u one can infer (v **implies** w).

In the locale L, identify now the operations $\wedge$, $\vee$ with the logical connectors **and**, **or** and interpret $u \leq v$ as "from u one can infer v". You conclude at once that the notation in Theorem 1.14 is the sensible one.

1.3 Localic toposes

Let us introduce a first time the word *topos*

17 **Definition 1.15** *The category of sheaves on a locale is called a* localic topos.

This section focuses on some typical properties of a topos.

18 **Proposition 1.16** *A localic topos is complete and limits of sheaves are computed pointwise.*

Sketch of the proof The terminal object is the constant sheaf on the singleton, represented by the top element 1 of the locale L (see Example 1.11). The case of products is trivial. Next consider two morphisms α, β of sheaves and their pointwise equalizer K.

$$\begin{tikzcd} K \arrow[r, tail, "\kappa"] & F \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & G \end{tikzcd}$$

Given a covering $u = \bigvee_{i\in I}$ in L, a compatible family $(a_i \in K(u_i))_{i\in I}$ admits a unique gluing $a \in F(u)$ because F is a sheaf. By naturality and since $a_i \in K(u_i)$, $\alpha_u(a)$ and $\beta_u(a)$ restrict as $\alpha_{u_i}(a_i) = \beta_{u_i}(a_i)$. Since G is a sheaf, this forces $\alpha_u(a) = \beta_u(a)$, thus $a \in K(u)$. $\square$

19 **Proposition 1.17** *A localic topos is Cartesian closed.*

Sketch of the proof Given two sheaves G, H on the locale L, we must find a sheaf H^G such that for every sheaf F

$$\mathsf{Nat}(F \times G, H) \cong \mathsf{Nat}(F, H^G).$$

Choosing $F = L(-, u)$ a representable sheaf (see Example 1.11), the Yoneda lemma forces the definition:

$$H^G(u) \cong \mathsf{Nat}\big(L(-,u), H^G\big) \cong \mathsf{Nat}\big(L(-,u) \times G, H\big) \cong \mathsf{Nat}\big(G|_u, H\big)$$

where $G|_u$ is the restriction of the sheaf G to the down segment $\downarrow u$, namely $G|_u(v) = G(v)$ for $v \leq u$ and is empty elsewhere. The rest is routine computation. $\square$

Let us pursue this first chapter with the key property of a topos. In the category of sets, every subset $S \subseteq A$ admits a characteristic mapping

$$\varphi\colon A \longrightarrow \{0,1\}, \quad a \mapsto 1 \ \text{ iff } \ a \in S.$$

The inclusion $\{1\} \rightarrowtail \{0,1\}$ – which we shall write simply as $v\colon \mathbf{1} \rightarrowtail \mathbf{2}$ – induces then by pullback a bijection between the morphisms $\varphi\colon A \longrightarrow \mathbf{2}$ and the subobjects $S \subseteq A$.

20 **Theorem 1.18** *Every localic topos admits a* subobject classifier, *that is, a monomorphism $t\colon \mathbf{1} \rightarrowtail \Omega$ inducing by pullback, for every sheaf F, a bijection between the morphisms $\varphi\colon F \longrightarrow \Omega$ and the subobjects of F.*

Sketch of the proof **1** is the terminal sheaf represented by $1 \in L$ (Example 1.11) and Ω is the sheaf in Example 1.12. The morphism t is defined by

$$t_u \colon \{\star\} \rightarrowtail {\downarrow} u, \quad \star \mapsto u;$$

it is of course a monomorphism. Pulling back a morphism $\varphi \colon F \longrightarrow \Omega$ along t yields thus a subobject of F. Conversely, given a subsheaf $S \rightarrowtail F$, define its characteristic morphism by

$$\varphi_u \colon F(u) \longrightarrow \Omega(u) = {\downarrow} u, \quad a \mapsto \bigvee \{ v \leq u \,|\, a|_v \in S(v) \}.$$

It is routine to observe that this yields a bijection. $\square$

The construction in Theorem 1.18 is worth a comment, which will take its full importance in Chapter 4. Consider a subsheaf $S \rightarrowtail F$ and an element $a \in F(u)$, for some $u \in L$. One could ask the question: *does a belong to S?*. Well, what do we mean by such a question? If it is *does a belong to $S(u)$*, the answer is simply "yes" or "no". But the situation is much more subtle since maybe $a \notin S(u)$, but a restricts to some element in $S(v)$ for some $v < u$. And as the construction in the proof of Theorem 1.18 shows, there exists a biggest such $v \in L$. So when we ask the question *does a belong to S?*, instead of answering simply true or false, we better provide a much more precise answer by pointing out the greatest element of L where this becomes the case. In this spirit, the morphism φ in the proof of Theorem 1.18 applies the element a on the "best truth value" of the formula $a \in S$. Thus Ω appears so as the *sheaf of truth values* of the theory of sheaves on L.

We shall investigate, in the more general context of Chapter 2, the existence of colimits in a localic topos and the universal construction of the sheaf associated with a presheaf.

1.4 An application to ring theory

In this section, by a *ring* we always mean a commutative ring with unit. The proofs of the various results that we mention are technically rather involved; an explicit treatment of the question can be found in sections 2.10 and 2.11 of [2].

Let us recall that a *prime ideal* of a ring is a proper ideal $I \lhd R$ satisfying

$$\forall r, s \in R \quad rs \in I \Rightarrow r \in I \ \mathsf{or} \ s \in I.$$

A ring admits always "enough" prime ideals, since given a prime ideal $I \lhd R$ and an element $r \in R$ such that for each $n \in \mathbb{N}$, $r^n \notin I$, there exists a prime ideal J containing I but not r.

Let us recall also that a ring S is *local* when

$$\forall s \in S \quad s \text{ is invertible } \mathsf{or} \ 1 - s \text{ is invertible.}$$

The notion of local ring is a kind of (weak) generalization of the notion of field, where the only non invertible element is $0 \ldots$ but of course $1 - 0$ is invertible. Another easy example is the set of those rational numbers which can be written as a fraction whose denominator is not divisible by some prime number p (this is a special case of a so-called *ring of valuation*). Another example of a local ring is the ring of formal series with n variables on a field K.

Given a prime ideal $J \lhd R$ of a ring, the ring of fractions

$$R_J = \left\{ \frac{r}{s} \middle| r \in R,\ s \in R,\ s \notin J \right\}$$

with the usual equality, sum and product of fractions, becomes a local ring.

Let us now canonically associate, with every ring R, a topological space and a sheaf on it.

21 **Definition 1.19** *The* spectrum *of a ring is the set* $\mathsf{Spec}(R)$ *of its prime ideals provided with the topology generated by the basic open subsets, for all* $r \in R$

$$\mathcal{O}_r = \{J | J \textit{ is a prime ideal},\ r \notin J\}.$$

By definition of a prime ideal J, one has at once

$$r \notin J \text{ and } s \notin J \Leftrightarrow rs \notin J$$

which means that $\mathcal{O}_{rs} = \mathcal{O}_r \cap \mathcal{O}_s$. Thus a general open subset of the spectrum is an arbitrary union of subsets of the form $\mathcal{O}_r$.

22 **Definition 1.20** *The* structural space *of a ring R is the mapping*

$$\gamma\colon \coprod_{J \in \mathsf{Spec}(R)} R_J \longrightarrow \mathsf{Spec}(R)$$

where the whole of each R_J is mapped on the single point J and the domain of γ is provided with the final topology for all mappings

$$\sigma^r_s\colon \mathcal{O}_r \longrightarrow \coprod_{J \in \mathsf{Spec}(R)} R_J, \quad J \mapsto \frac{r}{s} \in R_J, \quad r, s \in R.$$

As you can expect, since this example is presented here:

23 **Proposition 1.21** *The structural space of a ring is an étale mapping.* □

By Theorem 1.7, the structural mapping of the ring R corresponds to a sheaf Γ on $\mathsf{Spec}(R)$. Given an open subset $U \subseteq \mathsf{Spec}(R)$, $\Gamma(U)$ is the set of continuous sections of γ on U. Given such a section σ on U, we have thus

$$\forall J \in U \quad \big(\sigma(J) \in R_J \text{ is invertible or } 1 - \sigma(J) \in R_J \text{ is invertible}\big)$$

since each R_J is a local ring. But in fact a much stronger property holds.

24 **Theorem 1.22** *Consider a ring R, its structural space γ and the corresponding sheaf Γ as above.*

1. *The sheaf Γ is a sheaf of rings: that is, for each open subset $U \subseteq \mathsf{Spec}(R)$, $\Gamma(U)$ is a ring and each restriction mapping is a morphism of rings.*

2. *The ring R is isomorphic to $\Gamma\big(\mathsf{Spec}(R)\big)$, the ring of global sections of γ.*

3. *For every open subset $U \subseteq \mathsf{Spec}(R)$, every $\sigma \in \Gamma(U)$ viewed as a continuous section of γ on U and every $J \in U$, there exists an open neighborhood V of J in U such that*

$$\big(\forall I \in V\ \ \sigma(I) \in R_I \text{ is invertible}\big) \text{ or } \big(\forall I \in V\ \ 1 - \sigma(I) \in R_I \text{ is invertible}\big).$$

25 **Corollary 1.23** *Consider a ring R, its structural space γ and the corresponding sheaf Γ as above. For every element $r \in R$ viewed as a global continuous section σ of γ, there exists an open covering of $\mathsf{Spec}(R)$ such that, on each piece U of that covering,*

$$\sigma|_U \text{ is invertible or } (1 - \sigma)|_U \text{ is invertible.}$$

Theorem 4.10 will turn Corollary 1.23 into the statement that, in its topos of sheaves on its structural space, every ring becomes a local ring.

Chapter 2

Grothendieck toposes

2.1 Sheaves on a site

Let us first revisit the notion of sheaf on a locale L in more categorical terms. Let F be a presheaf on L. A compatible family $(a_i)_{i\in I}$ in F along a covering $u = \bigvee_{i\in I} u_i$ extends by restrictions to a compatible family along all the elements $v \in L$ smaller than some u_i; of course, both data are equivalent. But defining

$$R(v) = \{\star\} \ \ \mathsf{iff} \ \ \exists i \in I \ \ v \leq u_i$$

we get now a subpresheaf $R \rightarrowtail L(-, u)$ and giving the extended family is the same as giving a morphism of presheaves $\alpha\colon R \longrightarrow F$. Requiring the existence of a unique gluing $a \in F(u)$ is then equivalent, by the Yoneda lemma, to requiring that α extends uniquely to $L(-, u)$.

$$\begin{tikzcd}
R \arrow[r, tail] \arrow[d, "\alpha"'] & L(-,u) \arrow[dl, dotted, "\beta"] \\
F &
\end{tikzcd}$$

So, calling *covering* a subobject $R \rightarrowtail L(-u)$ such that

$$u = \bigvee \{v \big| R(v) = \{\star\}\}$$

the sheaf condition on a presheaf F is the so-called *orthogonality* condition to all the covering subobjects.[1]

One could be tempted to define a site as a small category $\mathcal{C}$ provided, for each object $C \in \mathcal{C}$, with a family of subobjects of $\mathcal{C}(-, C)$ chosen as the "covering ones". But if one expects to extend the properties of sheaves encountered in the case of a locale, the families of covering subobjects should mimic the properties of the covering families in a locale L:

1. For each $u \in L$, u covers u.

[1] In a category, an object X is said *orthogonal* to a morphism $f\colon A \longrightarrow B$ when each morphism $A \longrightarrow X$ factors uniquely through f.

2. If the u_i's cover u and $v \leq u$, then the $v \wedge u_i$'s cover v.

3. If the u_i's cover u and the $v_j \leq u$ are an arbitrary family whose restrictions cover each u_i, then the v_j's cover u.

26 **Definition 2.1** *Let $\mathcal{C}$ be a small category. Call* sieve *a subobject of a representable functor. A* Grothendieck topology *$\mathcal{T}$ on $\mathcal{C}$ consists in specifying, for each object $C \in \mathcal{C}$, a family $\mathcal{T}(C)$ of* covering sieves*, so that the following axioms are satisfied.*

1. *For each C, $\mathcal{C}(-, C)$ covers C.*

2. *If R covers C and $f \colon D \longrightarrow C$, then $\mathcal{C}(-, f)^{-1}(R)$ covers D.*

3. *Let R cover C and consider S, an arbitrary sieve on C. If for every $D \in \mathcal{C}$ and every $f \in R(D)$, $\mathcal{C}(-, f)^{-1}(S)$ covers D, then S covers D.*

A small category provided with a Grothendieck topology is called a site

27 **Definition 2.2** *Let $(\mathcal{C}, \mathcal{T})$ be a site.*
A presheaf *on $(\mathcal{C}, \mathcal{T})$ is a contravariant functor $\mathcal{C} \longrightarrow \mathsf{Set}$.*
A sheaf *on $(\mathcal{C}, \mathcal{T})$ is a presheaf F, orthogonal to every covering sieve.*
A morphism of sheaves or presheaves is a natural transformation between them.
The category of sheaves on a site is called a Grothendieck topos.

Everything has been done so that:

28 **Example 2.3** *Every localic topos is a Grothendieck topos.* □

Other trivial examples are given by:

29 **Example 2.4** *Every category of presheaves on a small category $\mathcal{C}$ is a Grothendieck topos.*

Sketch of the proof Choosing the identities on the representable functors as only covering sieves yields a Grothendieck topology. Every presheaf is then a sheaf. □

29a **Example 2.5** *The category of sets is a Grothendieck topos.*

Sketch of the proof The category of sets is that of presheaves on the terminal category $\mathbf{1}$. □

29b **Example 2.6** *Given a group G, the category of G-sets is a Grothendieck topos.*

Sketch of the proof Let us recall that a G-set is a set X provided with an action of G, so that

$$\forall x \in X \quad \forall g, g' \in G \quad x \cdot 1 = x, \quad x \cdot (gg') = (x \cdot g) \cdot g'.$$

Viewing the multiplicative group G as a one-object category admitting G and its multiplication as set of arrows, a G-set is just a presheaf. □

30 **Example 2.7** *Let $\mathcal{C}$ be a small category. Declaring all sieves covering yields a Grothendieck topology, for which the only sheaf is the constant presheaf on the singleton.*

Sketch of the proof Just because the empty presheaf is covering. $\square$

Observe that in Example 2.4, all representable functors are sheaves, while in Example 2.7, they are not in general. In fact one can prove that:

31 **Proposition 2.8** *The Grothendieck topologies on a small category, ordered by inclusion, constitute a locale.* $\square$

32 **Corollary 2.9** *Given a small category $\mathcal{C}$, there exists a biggest Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$ such that all representable functors are sheaves. It is called the* canonical topology *on $\mathcal{C}$.*

Sketch of the proof By Proposition 2.8, this is the supremum of all Grothendieck topologies for which the representable functors are sheaves. $\square$

2.2 The associated sheaf functor

Given a site $(\mathcal{C}, \mathcal{T})$, this section investigates the existence of the sheaf aF universally associated with a presheaf F. That is, we want to prove the existence of a left adjoint to the inclusion $\mathsf{Sh}(\mathcal{C}, \mathcal{T}) \subseteq \mathsf{Pr}(\mathcal{C})$ of the category of sheaves in that of presheaves.

Assuming that the problem is solved, given a covering sieve $R \rightarrowtail \mathcal{C}(-, C)$ and a morphism $f\colon R \longrightarrow F$, we get a unique factorization g as in the diagram

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d, "f"'] & \mathcal{C}(-,C) \arrow[d, dotted, "g"] \\ F \arrow[r] & aF \end{tikzcd}$$

where $F \longrightarrow aF$ is the unit of the adjunction. By the Yoneda lemma, giving g is giving an element of $aF(C)$: thus each f as above yields an element of $aF(C)$. This explains why we are interested in the following construction.

35 **Proposition 2.10** *Consider a site $(\mathcal{C}, \mathcal{T})$. For every presheaf F and every object $C \in \mathcal{C}$, define*

$$\alpha(F)(C) = \mathsf{colim}_{\,R \in \mathcal{T}(C)} \mathsf{Nat}(R, F).$$

This extends at once as a presheaf $\alpha(F)$ and further, as a functor

$$\alpha\colon \mathsf{Pr}(\mathcal{C}) \longrightarrow \mathsf{Pr}(\mathcal{C})$$

on the category of presheaves on $\mathcal{C}$. $\square$

34 **Lemma 2.11** *Given a site $\mathcal{C},\mathcal{T})$, the intersection of two covering sieves on an object C is still a covering sieve.*

Sketch of the proof This is a direct consequence of axiom 3 in Definition 2.1. □

34a **Corollary 2.12** *In the conditions of Proposition 2.10, the functor α preserves finite limits.*

Sketch of the proof By Lemma 2.11, the colimit in the definition of $\alpha(F)(C)$ is filtered. The result follows then from the commutation of finite limits with filtered colimits in Set, thus also in every category of presheaves. □

To clarify the language, let us introduce an intermediate notion: in Definition 2.2, we keep only the uniqueness condition in the orthogonality condition.

33 **Definition 2.13** *Let $(\mathcal{C},\mathcal{T})$ be a site. A* separated presheaf *on this site is a presheaf F such that, given a covering sieve $r\colon R \rightarrowtail \mathcal{C}(-,C)$ and a morphism $f\colon R \longrightarrow F$, there is at most one factorization of f through r.*

36 **Proposition 2.14** *In the situation of Proposition 2.10:*

1. *the presheaf $\alpha(F)$ is separated;*
2. *when F is separated, $\alpha(F)$ is a sheaf.*

Sketch of the proof See Section 3.3 of [2] for the involved details of the proof. □

As a (non trivial) corollary, we obtain the expected result:

37 **Theorem 2.15** *Let $(\mathcal{C},\mathcal{T})$ be a site. The category $\mathsf{Sh}(\mathcal{C},\mathcal{T})$ is a full reflective subcategory of the category $\mathsf{Pr}(\mathcal{C})$ of presheaves and the reflection, called the* associated sheaf functor*, preserves finite limits.*

Sketch of the proof Given a presheaf F, one defines $aF = \alpha\big(\alpha(F)\big)$. We obtain a natural transformation $\sigma_F\colon F \Rightarrow \alpha(F)$ by putting

$$\sigma_{F,C}\colon F(C) \longrightarrow \alpha(F)(C), \quad x \mapsto [x] \in \mathsf{Nat}\big(\mathcal{C}(-,C),F\big)$$

where $[x]$ is the equivalence class in the colimit of the morphism $\mathcal{C}(-,C) \longrightarrow F$ corresponding to the element $x \in F(C)$ by the Yoneda lemma. The unit of the adjunction is then given by the composite $\sigma_{\alpha(F)} \circ \sigma_F$. □

2.3 Limits and colimits in Grothendieck toposes

First of all:

38 **Proposition 2.16** *A Grothendieck topos is complete and cocomplete.*

Sketch of the proof Given a site $(\mathcal{C}, \mathcal{T})$, the category $\mathsf{Pr}(\mathcal{C})$ is complete and cocomplete: limits and colimits are computed pointwise. The category $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ is complete and cocomplete as a full reflexive subcategory of $\mathsf{Pr}(\mathcal{C})$. $\square$

39 **Proposition 2.17** *In a Grothendieck topos* $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$

1. *finite limits commute with filtered colimits;*
2. *colimits are universal;*[2]
3. *sums are disjoint.*[3]

Sketch of the proof All these properties hold in Set, thus in the topos $\mathsf{Pr}(\mathcal{C})$ of presheaves where limits and colimits are computed pointwise.

By Theorem 2.15, the category $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ is stable in $\mathsf{Pr}(\mathcal{C})$ under limits, while colimits are obtained by applying the associated sheaf functor to the corresponding colimit computed in $\mathsf{Pr}(\mathcal{C})$. This allows to conclude, since the associated sheaf functor preserves all colimits and finite limits (thus also in particular, monomorphisms and the initial object). $\square$

Let us recall that an epimorphism is *regular* when it is a coequalizer. In Set, this means just being surjective. In a topos $\mathsf{Pr}(\mathcal{C})$ of presheaves, this means therefore being pointwise surjective. The inclusion of sheaves in presheaves does not in general preserve colimits, thus a regular epimorphism of sheaves has no reason to be pointwise surjective.

40 **Proposition 2.18** *Every Grothendieck topos* $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ *is a regular and exact category.*

Sketch of the proof Let us recall that a finitely complete and cocomplete category is *regular* when regular epimorphisms are pullback stable. It is *exact* when moreover, every equivalence relation is *effective*, that is, is the kernel pair of its cokernel. This is the case in Set, thus in every topos of presheaves.

If $f\colon A \twoheadrightarrow B$ is a regular epimorphism of sheaves, factor it through its image in $\mathsf{Pr}(\mathcal{C})$

$$A \xrightarrow{p} \!\!\!\!\twoheadrightarrow I \rightarrowtail^{i} B.$$

We have $f = a(f) = a(i) \circ a(p)$; $a(i)$ is a monomorphism because the functor a preserves them (see Theorem 2.15), but is also a regular epimorphism since so is f. Thus $a(i)$ is an isomorphism and f, up to isomorphism, has the form $a(p)$ for a regular epimorphism p of presheaves and so is regular. The rest is routine. $\square$

[2] A colimit $A = \lim_i A_i$ is universal when pulling it back along whatever morphism $B \longrightarrow A$ yields another colimit cocone.

[3] The canonical morphisms $A_i \longrightarrow \coprod_i A_i$ are monomorphisms and the intersection of two of them is the initial object.

2.4 Closure operator and subobject classifier

In this section, we want to generalize to the case of a Grothendieck topos the properties of localic toposes exhibited in Section 1.3. We have already observed that a Grothendieck topos is both complete and cocomplete (see Proposition 2.16).

41 **Proposition 2.19** *Every Grothendieck topos is Cartesian closed.*

Sketch of the proof Let $(\mathcal{C}, \mathcal{T})$ be a site. The Cartesian closedness of the category $\mathsf{Pr}(\mathcal{T})$ of presheaves means the existence, given two presheaves G and H, of a presheaf H^G such that for every presheaf F

$$\mathsf{Nat}(F \times G, H) \cong \mathsf{Nat}(F, H^G).$$

Putting $F = \mathcal{C}(-, C)$, the Yoneda lemma indicates that necessarily

$$H^G(C) \cong \mathsf{Nat}\big(\mathcal{C}(-, C), H^G\big) \cong \mathsf{Nat}\big(\mathcal{C}(-, C) \times G, H\big).$$

It is routine to observe that putting

$$H^G(C) = \mathsf{Nat}\big(\mathcal{C}(-, C) \times G, H\big)$$

yields the expected result in the topos of presheaves.

To conclude in the case of sheaves, it suffices to observe that when G and H are sheaves, so is H^G. In fact, for H^G being a sheaf, it suffices that H be a sheaf. $\square$

The case of the subobject classifier is more involved. Let us begin with the easy case of presheaves.

43 **Proposition 2.20** *Every topos of presheaves admits a subobject classifier.*

Sketch of the proof Let $\mathcal{C}$ be a small category. If a subobject classifier $t\colon \mathbf{1} \rightarrowtail \Omega$ exists in $\mathsf{Pr}(\mathcal{C})$, one must have, by the Yoneda lemma and the definition of a subobject classifier

$$\Omega(C) \cong \mathsf{Nat}\big(\mathcal{C}(-, C), \Omega\big) \cong \{S | S \text{ is a subobject of } \mathcal{C}(-, C)\}.$$

It is routine to observe that the definition

$$\Omega(C) = \{S | S \text{ is a subobject of } \mathcal{C}(-, C)\}$$

yields the expected result. $\square$

46 **Definition 2.21** *Consider a site $(\mathcal{C}, \mathcal{T})$. Given a presheaf P and a subpresheaf $S \rightarrowtail P$, the subpresheaf $\overline{S} \rightarrowtail P$ defined by*

$$\overline{S}(C) = \big\{x \in P(C) \big| \exists R \in \mathcal{T}(C)\ \forall D\ \forall f \in R(D)\ P(f)(x) \in S(D)\big\}$$

is called the closure *of the subobject S for the topology $\mathcal{T}$.*

The closure of a subpresheaf $S \rightarrowtail P$ consist thus in adding to S all those elements of P which lie “locally” in S, that is, whose restrictions along all the morphisms of a covering sieve lie in S.

48 **Lemma 2.22** *Given a site $(\mathcal{C}, \mathcal{T})$, the corresponding closure operator on subsheaves is pullback stable.*

Sketch of the proof By routine computation. $\square$

47 **Proposition 2.23** *Let $(\mathcal{C}, \mathcal{T})$ be a site. The corresponding topos of sheaves admits the subobject classifier $t_{cl} \colon \mathbf{1} \rightarrowtail \Omega_{cl}$ where*

$$\Omega_{cl}(C) = \big\{ S \big| S \text{ is a closed subobject of } \mathcal{C}(-, C) \big\}$$

and $t_{cl} \colon \mathbf{1} \longrightarrow \Omega_{cl}$ is the factorization of t (see Proposition 2.20) through Ω_{cl}.

Sketch of the proof By Lemma 2.22, Ω_{cl} is a presheaf. It is routine to check that Ω_{cl} is a sheaf, but you can also find a detailed proof in Example 5.2.9 of [2]. To conclude, it remains essentially to observe that given a subsheaf $S \rightarrowtail F$ and its characteristic mapping $\varphi \colon F \longrightarrow \Omega$ in the topos of presheaves (see Proposition 2.20), φ factors through Ω_{cl}. □

Let us conclude this section with a link between Grothendieck toposes and locales.

42 **Proposition 2.24** *In a Grothendieck topos, the subobjects of every object constitute a locale.*

Sketch of the proof Consider a sheaf F. The pointwise intersection of two subsheaves of F remains a subsheaf. Next given a family $(S_i \rightarrowtail F)_{i \in I}$ of subobjects, the union of that family is the image factorization[4] of the corresponding factorization $\coprod_{i \in I} S_i \longrightarrow F$ through the coproduct. By Propositions 2.17 and 2.18, coproducts and image factorizations are preserved by pullbacks. This is in particular the case when pulling back along an arbitrary subobject $S \rightarrowtail F$. □

[4]In a regular category, every morphism factors as a regular epimorphism followed by a monomorphism

Chapter 3

Elementary toposes

3.1 The axioms for a topos

As we shall observe, amazingly enough, all main characteristic properties of Grothendieck toposes can be inferred from the following elementary axioms:

50 **Definition 3.1** *An* elementary topos *is a category $\mathcal{E}$ satisfying the following three axioms:*

1. *$\mathcal{E}$ has finite limits;*
2. *$\mathcal{E}$ is Cartesian closed;*
3. *$\mathcal{E}$ admits a subobject classifier $t\colon \mathbf{1} \rightarrowtail \Omega$.*

Lawvere's original definition of an elementary topos required also the existence of finite colimits. It has been later observed that this axiom was redundant, but the proof of this fact is very involved (see Section 5.7 in [2]).

60 **Theorem 3.2** *An elementary topos has finite colimits.*

Sketch of the proof The idea is to use the Beck monadicity criterion to prove that the functor

$$\Omega^{(-)}\colon \mathcal{E}^{op} \longrightarrow \mathcal{E}$$

is monadic. When this is done, since $\mathcal{E}$ has finite limits, so does its dual category $\mathcal{E}^{op}$; thus $\mathcal{E}$ has finite colimits. □

As observed in Sections 1.3 and 2.4:

51 **Example 3.3** *Every Grothendieck topos, thus in particular every localic topos, is an elementary topos.* □

To support the intuition, we shall often illustrate our results in the very basic case of the topos Set of sets (see Example 2.5).

51a **Example 3.4** *The structure of the category of sets as an elementary topos.*

Sketch of the proof Given sets G and H, H^G is just the set of mappings from G to H. On the other hand $\Omega = \{0,1\}$ and Ω^G is isomorphic to the set of subsets of G. □

But probably, Example 2.6 is also worth a closer look, since it provides a less trivial but nevertheless easy to handle situation.

55 **Example 3.5** *Given a group G, the structure of the category of G-sets as an elementary topos.*

Sketch of the proof Given two G-sets A and B, elementary computations show the the power object B^A is simply given by

$$B^A = \{f | f\colon A \to B \text{ is an ordinary mapping}\}$$

together with the action, for an element $g \in g$

$$f \cdot g\colon A \longrightarrow B, \quad a \mapsto f(a \cdot g^{-1}) \cdot g.$$

On the other hand the subobject classifier Ω is the set of sub-G-sets of G (see Proposition 2.20), the only representable functor. But since every element in G has an inverse, there are only two such subobjects: the empty subobject and G itself. So $\Omega = \{\emptyset, G\}$ is the two-point G-set.

Putting these results together, we obtain that Ω^A is the set of ordinary mappings $A \longrightarrow \mathbf{2}$, which is thus isomorphic to the set of ordinary subsets of A: ordinary subsets, not sub-G-sets. □

Observe also that, since only *finite* limits are required:

52 **Example 3.6** *The category of finite sets is an elementary topos.* □

and more generally

53 **Example 3.7** *The category of presheaves of finite sets on a finite category is an elementary topos.* □

But maybe more amazingly, without any finiteness condition on G:

54 **Example 3.8** *Given an arbitrary group G, the category of finite G-sets is an elementary topos.*

Sketch of the proof This follows at once from the constructions in Example 3.5. □

3.2 Some set theoretical notions in a topos

Let us observe at once that various usual set-theoretical notions translate at once in an elementary topos.

56 **Definition 3.9** *Given an object A of an elementary topos $\mathcal{E}$, the characteristic mapping of the diagonal of A is called the* equality *on A.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "\Delta_A"'] & \mathbf{1} \arrow[d, tail, "t"] \\ A \times A \arrow[r, "=_A"'] & \Omega \end{tikzcd}$$

In the case of sets, the subobject classifier is

$$t\colon \mathbf{1} \rightarrowtail \Omega = \{0,1\}, \quad \star \mapsto 1$$

which from now on we shall instead write

$$t\colon \{\mathsf{true}\} \rightarrowtail \{\mathsf{false}, \mathsf{true}\}.$$

In the situation of Definition 3.9, we have then, for two elements $a, b \in A$

$$\big(=_A (a,b) = \mathsf{true}\big) \;\; \mathsf{iff} \;\; \big(a = b\big).$$

57 **Definition 3.10** *Given an object A of an elementary topos $\mathcal{E}$, the morphism $=_A$ of Definition 3.9 corresponds by Cartesian closedness to a morphism*

$$\{\cdot\}_A\colon A \longrightarrow \Omega^A$$

which is called the singleton *on A.*

In the case of sets, Ω^A is the set of subsets of A and given an element $a \in A$, the morphism $\{\cdot\}_A$ maps precisely a on the singleton $\{a\}$.

58 **Definition 3.11** *Given an object A of an elementary topos $\mathcal{E}$, the identity on Ω^A corresponds by Cartesian closedness to a morphism*

$$\in_A\colon A \times \Omega^A \longrightarrow \Omega$$

which is called the membership relation *on A.*

Again in the case of sets and with the notation above, the identity on Ω^A corresponds by Cartesian closedness to the mapping

$$(a \in A, \varphi\colon A \to \Omega) \mapsto \big(\varphi(a) \in \Omega\big).$$

The morphism φ is the characteristic mapping of some subset $S \subseteq A$ and in terms of subobjects, instead of characteristic mappings, the description of $\in_A$ becomes

$$\in_A (a, S) = \mathsf{true} \;\; \mathsf{iff} \;\; a \in S.$$

Let us observe the behavior of these morphisms $=_A$, $\{\cdot\}_A$ and $\in_A$ in the more involved case of a localic topos.

59 **Example 3.12** *Given a locale L and a sheaf F on L,*

1. $(=_F)_u(a,b) = \bigvee\big\{v \leq u \big| a|_v = b|_v\big\}$;
2. $\{\cdot\}_u(a)$ *is the subsheaf of A constituted of all the restrictions of a at all lower levels $v \leq u$;*
3. $\in_A (a, S) = \bigvee\big\{v \leq u \big| a|_v \in S(v)\big\}$;

where $u, v \in L$, $a, b \in A(u)$ and $S \subseteq A$.

Once more, we observe that these expressions indicate "to which extent" the corresponding classical expressions are true, thus underline once more the "local character" of the logic of a topos.

3.3 The slice toposes

See Section 5.8 of [2] for detailed proofs of the various results of this section.

60a **Proposition 3.13** *Given an elementary topos $\mathcal{E}$ and an object $I \in \mathcal{E}$, the slice category $\mathcal{E}/I$ is still a topos.* □

The proof of Proposition 3.13 is quite involved, but the spirit of it is easily grasped in the case of the topos of sets.

65 **Example 3.14** *The topos Set/I of sets over I, which is equivalent to Set^I.*

Sketch of the proof Given a set I, the slice category Set/I of arrows over I can be equivalently seen as the category of I-families of sets: an arrow $p\colon A \longrightarrow I$ yields the family $\big(p^{-1}(i)\big)_{i\in I}$ and conversely, a family $(A_i)_{i\in I}$ of sets yields the set $A = \coprod_{i\in I} A_i$, with the obvious projection $p\colon A \longrightarrow I$ mapping the whole of A_i on $i \in I$. The slice category Set/I is thus equivalent to the category of I-families of sets, that is, to the power category Set^I. In that particular case of sets, this is trivially a topos, with the exponentiation and the subobject classifier defined pointwise as in Set, for every index $i \in I$. □

66 **Theorem 3.15** *Consider a morphism $f\colon I \longrightarrow J$ in an elementary topos $\mathcal{E}$. The pullback functor*

$$f^*\colon \mathcal{E}/J \longrightarrow \mathcal{E}/I$$

preserves exponentiation and the subobject classifier, and admits both a left adjoint Σ_f and a right adjoint π_f.

Sketch of the proof The existence of the left adjoint Σ_f is a general fact which has nothing to do with the topos structure: $\Sigma_f(p) = f \circ p$. The existence of the right adjoint π_f is a deep result. □

Again, the case of sets will throw an interesting light on the question.

67 **Example 3.16** *The functors f^*, Σ_f and π_f in the topos of sets.*

Sketch of the proof In terms of families of sets, the mapping $f\colon I \longrightarrow J$ yields the pullback functor

$$f^*\colon \mathsf{Set}/J \longrightarrow \mathsf{Set}/I, \quad (B_j)_{j\in J} \mapsto \big(B_{f(i)}\big)_{i\in I}$$

which acts just by re-indexing the families of sets. The preservation of the exponentiation and the subobject classifier follows then from their pointwise definition, as observed in Example 3.14.

It is routine to check that the left adjoint Σ_f of f^* is simply

$$\Sigma_f (A_i)_{i\in I} = \left(\coprod_{\{i|f(i)=j\}} A_i \right)_{j\in J}$$

while the right adjoint π_f is given by

$$\pi_f (A_i)_{i\in I} = \left(\prod_{\{i|f(i)=j\}} A_i \right)_{j\in J}.$$

The existence of Σ_f and π_f in the case of an elementary topos is thus some kind of existence of internal coproducts and products. □

The functors Σ_f and π_f will play an important role in developing the internal logic of a topos and in particular, the quantifiers. This can be guessed at once from the following proposition.

68 **Proposition 3.17** *In the category of sets, consider a projection of a binary product:* $p_B \colon A \times B \longrightarrow B$*. Consider further a subset* $s \colon S \rightarrowtail A \times B$*. Writing* Im *for the image of a mapping*

$$\begin{aligned} \mathsf{Im}\, \Sigma_{p_B}(s) &= \{b \in B | \exists a \in A \ (a,b) \in S\} \rightarrowtail B \\ \pi_{p_B}(s) &= \{b \in B | \forall a \in A \ (a,b) \in S\} \rightarrowtail B \end{aligned}$$

Sketch of the proof Since s is injective, $S_{(a,b)} = s^{-1}(a,b)$ is the singleton $\{(a,b)\}$ when $(a,b) \in S$ and is empty otherwise. □

In view of proposition 3.17, it is sensible to define:

68a **Definition 3.18** *In a topos, consider a projection of a product* $p \colon A \times B \longrightarrow B$*. Consider further a subobject* $s \colon S \rightarrowtail A \times B$*. We shall write*

- $\exists_p(S)$ *for the image of* $\Sigma_p(s)$*, as subobject of* B*;*
- $\forall_p(S)$ *for* $\pi_p(s)$*, as subobject of* B*.*

3.4 Exactness properties

This section extends to elementary toposes various properties encountered in the case of a Grothendieck topos (see Section 2.3).

69 **Proposition 3.19** *In an elementary topos, all existing colimits are universal.*

Sketch of the proof The pullback functors have right adjoints by Theorem 3.15, thus preserve colimits. □

70 **Proposition 3.20** *An elementary topos is a regular and exact category.*

Sketch of the proof Coequalizers exist by Theorem 3.2 and are universal by Proposition 3.19. Thus the topos is a regular category. Its exactness means that every equivalence relation is effective, that is, is the kernel pair of its coequalizer. That part of the proof is technically more involved; see Proposition 5.9.6 in [2]. □

72 **Corollary 3.21** *In an elementary topos, every monomorphism is regular and every epimorphism is regular.*

Sketch of the proof The subobject classifier $t \colon \mathbf{1} \rightarrowtail \Omega$ admits the trivial retraction $r \colon \Omega \longrightarrow \mathbf{1}$, thus is a regular monomorphism: the equalizer of the pair $(\mathsf{id}_\Omega, t \circ r)$. Every monomorphism is then regular, as pullback of t along its characteristic morphism.

By Proposition 3.20, an epimorphism f factors as $f = i \circ p$, with p a regular epimorphism and i a monomorphism. Since f is an epimorphism, so is i. But i is also a regular monomorphism, thus an isomorphism. □

74 **Proposition 3.22** *In an elementary topos*

1. *The initial object* $\mathbf{0}$ *is* strict*;*
2. $\mathbf{0}$ *is a subobject of every object;*
3. *the canonical morphism* $A \longrightarrow A \coprod B$ *of a coproduct is a monomorphism;*
4. *finite coproducts are disjoint;*
5. *finite unions of subobjects exist.*

Sketch of the proof The strictness of $\mathbf{0}$ means that every morphism $A \longrightarrow \mathbf{0}$ is an isomorphism. This is the case by universality of the empty colimit (see Proposition 3.19). As a consequence, given any two morphisms $u, v\colon A \rightrightarrows \mathbf{0}$, A is an initial object and $u = v$, so that every morphism with domain $\mathbf{0}$ is a monomorphism.

One can prove (see Proposition 5.9.10 in [2]) that in a topos, the pushout of a monomorphism remains a monomorphism and the corresponding pushout square is also a pullback. But a coproduct $A \coprod B$ is the pushout of these two objects over $\mathbf{0}$. The morphisms of the coproduct are thus monomorphisms and the pushout square is also a pullback: this is the so-called *disjointness of coproducts*.

The union of two subobjects $R \rightarrowtail A$ and $S \rightarrowtail A$ is the image of the corresponding factorization $R \coprod S \longrightarrow A$. □

3.5 Heyting algebras in a topos

Let us introduce the notion of a *Heyting algebra*, which is closely related to that of a locale.

75 **Definition 3.23** *A* Heyting algebra *is a lattice with top and bottom element, in which for every two elements* s*,* t*, there exists an element* $s \Rightarrow t$ *such that for each element* r

$$r \wedge s \leq t \ \text{ iff } \ r \leq s \Rightarrow t.$$

76 **Proposition 3.24** *The locales are exactly the complete Heyting algebras.*

Sketch of the proof By Theorem 1.14, every locale is a Heyting algebra. Conversely in a Heyting algebra viewed as a category, $- \wedge s$ admits the right adjoint $s \Rightarrow -$, thus preserves all existing joins. □

76a **Proposition 3.25** *Every Boolean algebra is a Heyting algebra.*

Sketch of the proof Simply define $(s \Rightarrow t) = \complement s \vee t$. □

Complements do not exist in a Heyting algebra, but a weaker property holds:

13 **Proposition 3.26** *In a Heyting algebra* H*, every element* u *has a* pseudo-complement*, that is, a greatest element* $\neg u$ *whose meet with* u *is the bottom element* 0.

Sketch of the proof In Definition 3.23, simply put $\neg u = (u \Rightarrow 0)$. □

14 **Example 3.27** *In the locale of open subsets of a topological space, $\neg U$ is the interior of the set-complement of U.* □

77 **Theorem 3.28** *In an elementary topos, the subobjects of every object constitute a Heyting algebra.*

Sketch of the proof Given two subobjects $\sigma\colon S \rightarrowtail A$ and $\tau\colon T \rightarrowtail A$, it remains to prove the existence of the subobject $(S \Rightarrow T) \rightarrowtail A$. Writing φ_R for the characteristic morphism of a subobject $\rho\colon R \rightarrowtail A$, $S \Rightarrow T$ is defined as the following equalizer

$$(S \Rightarrow T) \rightarrowtail A \overset{\varphi_{S\cap T}}{\underset{\varphi_S}{\rightrightarrows}} \Omega.$$

By definition of an equalizer, we have

$$\begin{array}{rll} R \subseteq (S \Rightarrow T) & \text{iff} & \varphi_S \circ \rho = \varphi_{S\cap T} \circ \rho \\ & \text{iff} & R \cap S = R \cap (S \cap T) \\ & \text{iff} & R \cap S \subseteq T. \end{array}$$

Let us recall that in the case of sets, given subobjects S and T of a set A, $(S \Rightarrow T) = \complement S \cup T$ (see Example 1.13). But more interestingly, avoiding to use the Boolean algebra structure,

$$\begin{array}{rcl} (S \Rightarrow T) & = & \big\{a \in A \big| \{a\} \subseteq (S \Rightarrow T)\big\} \\ & = & \big\{a \in A \big| \{a\} \cap S \subseteq T\big\} \\ & = & \big\{a \in A \big| a \in S \text{ implies } a \in T\big\} \end{array}$$

which justifies further the notation. □

It is well-know that a notion like that of a Heyting algebra H can be internalized in every category $\mathcal{C}$ with finite limits:

- giving the top and bottom elements is giving morphisms $\mathbf{1} \longrightarrow H$;
- giving the operations $\wedge$, $\vee$, $\Rightarrow$ is giving morphisms $H \times H \longrightarrow H$.

The internal poset structure of H is defined as the equalizer

$$\leq_H \rightarrowtail H \times H \overset{\wedge}{\underset{p_1}{\rightrightarrows}} H.$$

One translates then the set theoretical axioms for a Heyting algebra via equalities or factorizations of arrows constructed from the given ones. For example

$$(r \wedge s) \leq t \ \text{ iff } \ r \leq (s \Rightarrow t)$$

can be translated via the existence of the two canonical natural transformations of the adjunction $(- \wedge s) \dashv (s \Rightarrow -)$:

$$(s \Rightarrow t) \wedge s \leq t, \quad r \leq \big(s \Rightarrow (r \wedge s)\big).$$

78 **Theorem 3.29** *The object Ω of an elementary topos $\mathcal{E}$ is provided with the structure of an internal Heyting algebra. For every object $A \in \mathcal{E}$, this induces by composition a Heyting algebra structure on the set $\mathcal{E}(A, \Omega)$ of morphisms; in terms of corresponding subobjects of A, this is the Heyting algebra structure of Theorem 3.28.*

Sketch of the proof The various ingredients for an internal Heyting algebra are defined as follows:

- the top element 1 is the subobject classifier $t\colon \mathbf{1} \rightarrowtail \Omega$; ($t$ for "true");
- the bottom element 0 is the characteristic morphism of the zero subobject $\mathbf{0} \rightarrowtail \mathbf{1}$; it is generally written $f\colon \mathbf{1} \longrightarrow \Omega$ (f for "false");
- $\wedge\colon \Omega \times \Omega \longrightarrow \Omega$ is the characteristic morphism of $\Delta_\Omega\colon \Omega \rightarrowtail \Omega \times \Omega$, the diagonal of Ω;
- $\vee\colon \Omega \times \Omega \longrightarrow \Omega$ is the characteristic morphism of the union of the two subobjects $t \times \mathsf{id}_\Omega\colon \mathbf{1} \times \Omega \longrightarrow \Omega \times \Omega$ and $\mathsf{id}_\Omega \times t\colon \Omega \times \mathbf{1} \longrightarrow \Omega$;
- $\Rightarrow\colon \Omega \times \Omega \longrightarrow \Omega$ is the characteristic morphism of the poset structure subobject $\leq_\Omega \rightarrowtail \Omega \times \Omega$ (see above);
- $\neg\colon \Omega \longrightarrow \Omega$, that is $(\bullet \Rightarrow 0)$ (see Corollary 3.26), is then the characteristic morphism of $f\colon \mathbf{1} \longrightarrow \Omega$.

The rest is straightforward calculation. $\square$

A last comment. When we think Ω as an object of *truth values* (see the comment after Theorem1.18), the internal poset structure of Ω "coincides" thus with the implication, as the definition of $\Rightarrow$ in the proof of Theorem 3.29 indicates.

Chapter 4

Internal logic of a topos

In this chapter "topos" always means "elementary topos", except otherwise specified.

4.1 The language of a topos

80 Suppose you want to study the field of real numbers. You will have to handle "actual numbers" like 5, $\frac{2}{3}$, π, $\sqrt{2}$, and so on. We call these *constants of type* $\mathbb{R}$: there are thus as many such constants as real numbers. But you will also have to handle formulæ like

$$a \times (b + c) = (a \times b) + (a \times c)$$

where a, b, c stand now for arbitrary, unspecified real numbers. We call a, b, c *variables of type* $\mathbb{R}$. Since a formula which you can write is a finite sequence of symbols, you only need each time a finite (possibly very big) number of such variables ... thus it suffices to give yourself a denumerable set of variables of type $\mathbb{R}$ in order to be able to write down all possible formulæ. The number of variables has thus nothing to do with the number of elements of $\mathbb{R}$. And notice that if you wanted to study instead the singleton ... you would already need two variables in order to express that the singleton has only one element

$$\forall x \; \forall y \; x = y.$$

81 **Definition 4.1** *The* language of a topos $\mathcal{E}$ *consists in giving, for every object* $A \in \mathcal{E}$

- *a formal symbol, called a* constant of type A*, for every arrow* $\mathbf{1} \longrightarrow A$*;*
- *a denumerable set of formal symbols, called the* variables of type A*.*

From these ingredients, for every object $A \in \mathcal{E}$, one constructs inductively, as usual, the terms and formulæ. Again as usual, the notion of free variable is that of a variable which is not bounded by a quantifier $\exists$ or $\forall$. Of course a term or a formula with free variables $a_1, \ldots, a_n$ can always be seen as a term or a formula with a bigger set of variables ... where the additional variables do not appear. In this spirit, there is generally no restriction in considering that two terms or formulæ have the same set of free variables. A precise discussion on the effect of handling such "ghost variables" can be found in Section 6.4 of [2].

It is probably useful to recall that in a topos (see Definition 3.1), the "constants" of type Ω^A, that is, the morphisms $\mathbf{1}\longrightarrow\Omega^A$, are in bijection with the morphisms $A\longrightarrow\Omega$ and thus further with the subobjects of A. Thus Ω^A should be thought as the "object of subobjects of A".

82 **Definition 4.2** *In a topos $\mathcal{E}$, the terms of the internal language are the formal expressions defined inductively by:*

1. *the constants of type A are terms of type A;*
2. *the variables of type A are terms of type A;*
3. *if τ is a term of type A and $f\colon A\longrightarrow B$ is a morphism in $\mathcal{E}$, $f(\tau)$ is a term of type B;*
4. *if $\tau_1,\dots,\tau_n$ are terms of respective types $A_1,\dots,A_n$, then $(\tau_1,\dots,\tau_n)$ is a term of type $A_1\times\cdots\times A_n$;*
5. *if τ is a term of type A with free variables $a_1,\dots,a_n$ of respective types $A_1,\dots,A_n$; if $\sigma_1,\dots,\sigma_n$ are terms of respective types $A_1,\dots,A_n$, not containing any bound variable of τ, then $\tau(\sigma_1,\dots,\sigma_n)$ remains a term of type A;*
6. *if φ is a formula with free variables $a_1,\dots,a_n,b_1,\dots,b_m$ of respective types $A_1,\dots,A_n,B_1,\dots,B_m$*
$$\big\{(a_1,\dots,a_n)\big|\varphi(a_1,\dots,a_n,b_1,\dots,b_m)\big\}$$
is a term of type $\Omega^{A_1\times\ldots\times A_n}$.

83 **Definition 4.3** *In a topos $\mathcal{E}$, the formulæ of the internal language are the formal expressions defined inductively by:.*

1. *the symbols* true *and* false *are formulæ;*
2. *if τ and σ are terms of type A, then $\tau=\sigma$ is a formula.*
3. *if τ is a term of type A and Σ is a term of type Ω^A, then $\tau\in\Sigma$ is a formula;*
4. *if φ is a formula, then $\neg\varphi$ is a formula;*
5. *if φ and ψ are formulæ, then $\varphi\wedge\psi$, $\varphi\vee\psi$ and $\varphi\Rightarrow\psi$ are formulæ;*
6. *if φ is a formula with free variables $a,b_1,\dots,b_n$ of respective types $A,B_1,\dots,B_n$, then*
$$\exists a\ \varphi(a,b_1,\dots,b_n),\quad \forall a\ \varphi(a,b_1,\dots,b_n)$$
are formulæ with free variables $b_1,\dots,b_n$;
7. *if φ is a formula with free variables $a_1,\dots,a_n$ of respective types $A_1,\dots,A_n$ and $\sigma_1,\dots,\sigma_n$ are terms of respective types $A_1,\dots,A_n$ with the same free variables $b_1,\dots,b_m$ of respective types $B_1,\dots,B_m$, then*
$$\varphi\big(\sigma_1(b_1,\dots,b_m),\dots,\sigma_n(b_1,\dots,b_m)\big)$$
is a formula with free variables $b_1,\dots,b_m$.

Of course as usual, $\exists! x\, \varphi(x)$ is an abbreviation for

$$\big(\exists x\, \varphi(x)\big) \wedge \big((\varphi(y) \wedge \varphi(z)) \Rightarrow (y = z)\big).$$

In other words, this section can be summarized by saying that the language of a topos mimics exactly the usual language of set theory.

4.2 Interpretation of terms and formulæ

Let us go back to our example of real numbers. Given a rational number a and a natural number b, we can construct the real number $b^{a\pi}$. In the language of Section 4.1, $b^{a\pi}$ is thus a term of type $\mathbb{R}$ with two variables a, b of respective types $\mathbb{Q}$ and $\mathbb{N}$. Such a term induces thus a mapping

$$\mathbb{Q} \times \mathbb{N} \longrightarrow \mathbb{R}, \quad (a, b) \mapsto b^{a\pi}.$$

That mapping is what we shall call the *realization of the term $b^{a\pi}$*.

The construction which will follow intends to associate, with every term τ of type A with free variables $a_1, \ldots, a_n$ of respective types $A_1, \ldots, A_n$, a "realization" morphism

$$\ulcorner \tau \urcorner \colon A_1 \times \cdots \times A_n \longrightarrow A.$$

Next consider, for three variables of type $\mathbb{R}$, the formula $a + b = c$. This yields at once a mapping

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$

which maps each concrete triple (a, b, c) of real numbers on the "truth value" of $a + b = c$. This mapping is what we shall call the *truth table* of the formula $a + b = c$. Let us recall that in the topos of sets, $\{\mathsf{true}, \mathsf{false}\}$ is precisely Ω, the subobject classifier..

The comment following Theorem 1.18 points out that, in a localic topos, the object Ω is worth being thought as the *object of truth values*. The construction which will follows extends this idea to an elementary topos. We shall associate, with every formula φ with free variables $a_1, \ldots, a_n$ of respective types $A_1, \ldots, A_n$, a "truth table" morphism

$$\ulcorner \varphi \urcorner \colon A_1 \times \cdots \times A_n \longrightarrow \Omega.$$

The subobject classified by this morphism will be written

$$[\varphi] = \big\{(a_1, \ldots, a_n) \big| \varphi(a_1, \ldots, a_n)\big\} \rightarrowtail A_1 \times \cdots \times A_n.$$

Again the definitions proceed inductively. For short, we use without recalling them the notation and numeration of Definitions 4.2 and 4.3.

84 **Definition 4.4** *In a topos $\mathcal{E}$, with the notation and the numeration of Definition 4.2, the* realization $\ulcorner \tau \urcorner$ *of a term τ is defined inductively by:*

1. *the realization of a constant is the constant itself;*

2. *the realization of a variable of type A is the identity on A;*

3. $\ulcorner f(\tau)\urcorner = f \circ \ulcorner\tau\urcorner$;

4. $\ulcorner(\tau_1, \ldots, \tau_n)\urcorner = (\ulcorner\tau_1\urcorner, \ldots, \ulcorner\tau_n\urcorner)$

5. $\ulcorner\tau(\sigma_1, \ldots, \sigma_n)\urcorner = \ulcorner\tau\urcorner \circ (\ulcorner\sigma_1\urcorner, \ldots, \ulcorner\sigma_n\urcorner)$;

6. $\ulcorner\{(a_1, \ldots, a_n)|\varphi\}\urcorner$ *is the morphism* $B_1 \times \cdots \times B_m \longrightarrow \Omega^{A_1 \times \cdots \times A_n}$ *corresponding to* $\ulcorner\varphi\urcorner$ *by Cartesian closedness.*

85 **Definition 4.5** *In a topos $\mathcal{E}$, with the notation and the numeration of Definition 4.3 and using the various constructions in Definition 3.9, Definition 3.11, Theorem 3.29, Definition 3.18, the* truth table $\ulcorner\varphi\urcorner$ *of a formula φ is defined inductively by:*

1. $\ulcorner\mathsf{true}\urcorner = t$ *and* $\ulcorner\mathsf{false}\urcorner = f$;

2. $\ulcorner\tau = \sigma\urcorner = (=_A) \circ (\ulcorner\tau\urcorner, \ulcorner\sigma\urcorner)$;

3. $\ulcorner\tau \in \Sigma\urcorner = \in_A \circ(\ulcorner\tau\urcorner, \ulcorner\Sigma\urcorner)$;

4. $\ulcorner\neg\varphi\urcorner = \neg \circ \ulcorner\varphi\urcorner$;

5. $\ulcorner\varphi \wedge \psi\urcorner = \wedge \circ (\ulcorner\varphi\urcorner), \ulcorner\psi\urcorner)$,
   $\ulcorner\varphi \vee \psi\urcorner = \vee \circ (\ulcorner\varphi\urcorner), \ulcorner\psi\urcorner)$,
   $\ulcorner\varphi \Rightarrow \psi\urcorner = (\Rightarrow) \circ (\ulcorner\varphi\urcorner), \ulcorner\psi\urcorner)$;

6. *writing* $p\colon A \times B_1 \times \cdots \times B_n \longrightarrow B_1 \times \cdots \times B_n$ *for the first projection and* $S \rightarrowtail A \times B_1 \times \cdots \times B_n$ *for the subobject classified by* $\ulcorner\varphi\urcorner$,

$$\ulcorner\exists a\ \varphi(a, b_1, \ldots, b_n)\urcorner = \exists_p(S), \quad \ulcorner\forall a\ \varphi(a, b_1, \ldots, b_n)\urcorner = \forall_p(S);$$

7. $\ulcorner\varphi(\sigma_1, \ldots, \sigma_n)\urcorner = \ulcorner\varphi\urcorner \circ (\ulcorner\sigma_1\urcorner, \ldots, \ulcorner\sigma_n\urcorner)$.

Of course true and false are formulæ without free variable. If you view them as formulæ with a (non appearing) free variable a of type A, the corresponding truth table are then

$$A \longrightarrow \mathbf{1} \xrightarrow{\ t\ } \Omega, \quad A \longrightarrow \mathbf{1} \xrightarrow{\ f\ } \Omega.$$

Let us conclude this section with an obvious but useful observation.

86 **Proposition 4.6** *In a topos $\mathcal{E}$, consider two formulæ φ, ψ with the same free variables $a_1, \ldots, a_n$ of respective types $A_1, \ldots, A_n$. Write $[\varphi]$ and $[\psi]$ for the subobjects of $A_1 \times \cdots \times A_n$ classified by $\ulcorner\varphi\urcorner$ and $\ulcorner\psi\urcorner$. Then the subobjects of $A_1 \times \cdots \times A_n$ classified by*

$$\ulcorner\mathsf{true}\urcorner, \quad \ulcorner\mathsf{false}\urcorner, \quad \ulcorner\varphi \wedge \psi\urcorner, \quad \ulcorner\varphi \vee \psi\urcorner, \quad \ulcorner\varphi \Rightarrow \psi\urcorner, \quad \ulcorner\neg\varphi\urcorner$$

are simply

$$A_1 \times \cdots \times A_n, \quad 0, \quad [\varphi] \wedge [\psi], \quad [\varphi] \vee [\psi], \quad [\varphi] \Rightarrow [\psi], \quad \neg[\varphi]$$

in the Heyting algebra of subobjects of $A_1 \times \cdots \times A_n$.

Sketch of the proof Routine computations from the definitions. □

Proposition 4.6 suggests at once that in the case of Grothendieck toposes, arbitrary conjunctions and disjunctions can be handled in the internal logic of the topos. Indeed, a Grothendieck topos is complete and cocomplete (see Proposition 2.16); in particular, arbitrary intersections and unions of subobjects exist. Consider then an arbitrary family $(\varphi_i)_{i\in I}$ of formulæ with the same free variables, their corresponding truth tables $(\ulcorner\varphi_i\urcorner)_{i\in I}$ and the corresponding classified subobjects $([\varphi_i])_{i\in I}$. The characteristic mappings of the subobjects $\bigcap_{i\in I}[\varphi_i]$ and $\bigcup_{i\in I}[\varphi_i]$ are then chosen as the truth tables of the formulæ $\bigwedge_{i\in I}\varphi_i$ and $\bigvee_{i\in I}\varphi_i$.

4.3 Propositional calculus in a topos

We have now to explain what it means for a formula to be *true* and to infer the corresponding rules valid in the internal logic of a topos.

87 **Definition 4.7** *In a topos $\mathcal{E}$, let φ be a formula with variables $a_1,\ldots,a_n$ of respective types $A_1,\ldots,A_n$. We shall say that this formula is* true *and we shall write $\models\varphi$ when $\ulcorner\varphi\urcorner=\ulcorner\mathsf{true}\urcorner$, that is, equivalently, when the subobject classified by $\ulcorner\varphi\urcorner$ is $A_1\times\cdots\times A_n$ itself.*

88 **Theorem 4.8** *In a topos $\mathcal{E}$, all the rules of intuitionistic propositional calculus hold. More explicitly, consider three formulæ φ, ψ, θ with the same free variables. The following properties hold.*

$$
\begin{array}{rl}
(P1) & \models \varphi\Rightarrow(\psi\Rightarrow\varphi)\\
(P2) & \models \big(\varphi\Rightarrow(\psi\Rightarrow\theta)\big)\Rightarrow\big((\varphi\Rightarrow\psi)\Rightarrow(\varphi\Rightarrow\theta)\big)\\
(P3) & \models \varphi\Rightarrow\big(\psi\Rightarrow(\varphi\wedge\psi)\big)\\
(P4) & \models \varphi\wedge\psi\Rightarrow\varphi\\
(P5) & \models \varphi\wedge\psi\Rightarrow\psi\\
(P6) & \models \varphi\Rightarrow(\varphi\vee\psi)\\
(P7) & \models \psi\Rightarrow(\varphi\vee\psi)\\
(P8) & \models (\varphi\Rightarrow\theta)\Rightarrow\big((\psi\Rightarrow\theta)\Rightarrow((\varphi\vee\psi)\Rightarrow\theta)\big)\\
(P9) & \models (\varphi\Rightarrow\psi)\Rightarrow\big((\varphi\Rightarrow\neg\psi)\Rightarrow\neg\varphi\big)\\
(P10) & \models \neg\varphi\Rightarrow(\varphi\Rightarrow\psi)\\
(P11) & \textit{If } \models\varphi \textit{ and } \models\varphi\Rightarrow\psi \textit{ then } \models\psi \textit{ (Modus Ponens)}
\end{array}
$$

Sketch of the proof By Definition 4.7 and Proposition 4.6, it suffices to prove th corresponding properties in the Heyting algebra of subobjects of $A_1\times\cdots\times A_n$. We must thus prove that the subobject of $A_1\times\cdots\times A_n$ classified by the truth value of each of these formulæ is $A_1\times\cdots\times A_n$ itself.

In fact, *(P1)* to *(P11)* hold in every Heyting algebra (see Definition 3.23). For example if a, b are two elements in a Heyting algebra H, proving *(P1)* is proving

$$1=\big(a\Rightarrow(b\Rightarrow a)\big).$$

Of course this is equivalent to proving

$$1\leq\big(a\Rightarrow(b\Rightarrow a)\big).$$

that is

$$a=1\wedge a\leq b\Rightarrow a.$$

This is further equivalent to proving

$$a \wedge b \leq a$$

which is obvious. □

The following proposition underlines a first significant difference with classical logic.

89 **Proposition 4.9** *In a topos $\mathcal{E}$, let φ and ψ be two formulæ with the same free variables.*

- *If $\models \varphi \wedge \psi$, then $\models \varphi$ and $\models \psi$.*
- *But if $\models \varphi \vee \psi$, one does not have in general $\models \varphi$ or $\models \psi$.*

Sketch of the proof As in the proof of Proposition 4.8, it suffices to consider the situation in an arbitrary Heyting algebra H. Given $a, b \in H$, of course $a \wedge b = 1$ forces $a = 1$ and $b = 1$. But trivially also, $a \vee b = 1$ does not imply that one of the two elements is equal to 1. □

Let us further illustrate this difference with classical logic by proving that, choosing to work in the adequate topos ... every ring becomes a local ring.

90 **Theorem 4.10** *Consider a commutative ring with unit R and the corresponding sheaf Γ in the topos of sheaves on the structural space γ of R (see Corollary 1.23). In the internal logic of this topos, the ring Γ is a local ring.*

Sketch of the proof By Theorem 1.22 and with the notation of its point 3, the truth value of "σ *is invertible*" is the biggest open subset $V \subseteq U$ on which it is the case; analogously, the truth value of "$1 - \sigma$ *is invertible*" is the biggest open subset $W \subseteq U$ where it is the case. By Definition 4.5, the truth value of

$$(\sigma \text{ is invertible}) \text{ or } (1 - \sigma \text{ is invertible})$$

is then $V \cup W = U$. So the truth table of that formula is precisely that of true (see Theorem 1.18). □

This example shows that in the internal logic of a topos, a statement like $\models \varphi \vee \psi$ is somehow only a *local* statement. You do not have $\models \varphi$ or $\models \psi$ (see Proposition 4.9), but in this case of sheaves on a topological space, each point admits a neighborhood on which you have $\models \varphi$ or $\models \psi$.

4.4 Predicate calculus in a topos

We consider now the additional logical rules involving quantifiers.

91 **Theorem 4.11** *In a topos $\mathcal{E}$, all the rules of intuitionistic predicate calculus hold. More explicitly, consider two formulæ φ, ψ with the same free variables. Consider further a term τ. The following properties hold.*

(P12) $\models \big(\forall x(\varphi \Rightarrow \psi)\big) \Rightarrow \big((\forall x\, \varphi) \Rightarrow (\forall x\, \psi)\big)$
(P13) $\models \big(\forall x(\varphi \Rightarrow \psi)\big) \Rightarrow \big((\exists x\, \varphi) \Rightarrow (\exists x\, \psi)\big)$
(P14) $\models \varphi \Rightarrow (\forall x\, \varphi)$ *when x is not a free variable of φ*
(P15) $\models (\exists x\, \varphi) \Rightarrow \varphi$ *when x is not a free variable of φ*
(P16) $\models (\forall x\, \varphi) \Rightarrow \varphi(\tau)$ *when τ does not contain any bound variable of φ and $\varphi(\tau)$ is the result of replacing x by τ in φ*
(P17) $\models \varphi(\tau) \Rightarrow (\exists x\, \varphi)$ *when τ does not contain any bound variable of φ and $\varphi(\tau)$ is the result of replacing x by τ in φ*
(P18) *If $\models \varphi$ then $\models \forall x\, \varphi$*

Sketch of the proof To simplify the notation, imagine that φ and ψ have the free variables x, a of respective types X, A. The formula in *(P12)* has the free variable a, thus proving *(T12)* reduces to proving the corresponding result in the Heyting algebra of subobjects of A. We consider the projection $p\colon X \times A \longrightarrow A$. Going back to Theorem 3.15, we must prove that

$$A = \Big[\big(\forall x(\varphi \Rightarrow \psi)\big) \Rightarrow \big((\forall x\, \varphi) \Rightarrow (\forall x\, \psi)\big)\Big]$$

which is the same as

$$A \leq \big[\forall x(\varphi \Rightarrow \psi)\big] \Rightarrow \big([\forall x\, \varphi] \Rightarrow [\forall x\, \psi]\big)$$

By definition of $\Rightarrow$ in a Heyting algebra, this reduces to

$$\big[\forall x(\varphi \Rightarrow \psi)\big] = A \wedge \big[\forall x(\varphi \Rightarrow \psi)\big] \leq \big([\forall x\, \varphi] \Rightarrow [\forall x\, \psi]\big)$$

and further to

$$\big[\forall x(\varphi \Rightarrow \psi)\big] \wedge [\forall x\, \varphi] \leq [\forall x\, \psi].$$

Now using the adjunction $p^{-1} \dashv \pi_p$ of Theorem 3.15, this is still equivalent to

$$p^{-1}\Big(\big[\forall x(\varphi \Rightarrow \psi)\big] \wedge [\forall x\, \varphi]\Big) \leq [\psi]$$

and thus to

$$p^{-1}\big[\forall x(\varphi \Rightarrow \psi)\big] \wedge p^{-1}[\forall x\, \varphi] \leq [\psi].$$

The counit of the adjunction $p^{-1} \dashv \pi_p$ indicates that $p^{-1}\forall x(S) \leq S$ for every subobject S of A. Therefore

$$p^{-1}\big[\forall x(\varphi \Rightarrow \psi)\big] \wedge p^{-1}[\forall x\, \varphi] \leq [\varphi \Rightarrow \psi] \wedge [\varphi] = \big([\varphi] \Rightarrow [\psi]\big) \wedge [\varphi] \leq [\psi].$$

The other properties are proved in an analogous way. □

Let us conclude this section with another observation on the "local character" of the logic of a topos: this time of the existential quantifier. With the notation of the proof of Theorem 4.11, $[\exists x\, \varphi]$ is the image of $[\varphi]$ under p:

$$\begin{tikzcd} [\varphi] \arrow[r, two heads, "q"] \arrow[d, tail] & {[\exists x\,\varphi]} \arrow[d, tail] \\ X \times A \arrow[r, "p"'] & A \end{tikzcd}$$

(see Definition 3.18). First observe that p is generally not an epimorphism: this is already the case in the topos of sets, when X is empty. And in a localic topos, the same kind of phenomena can appear when the sheaf X is empty at some levels.

But our point is different. By definition of an image, q is an epimorphism. Consider again the case of a localic topos. An epimorphism is generally not surjective at each level. Indeed, consider a non-compatible family $(x_i \in X(u_i))_{i\in I}$ and a compatible family $(a_i \in A(u_i))_{i\in I}$, with gluing $a \in A(u)$, where $u = \bigvee u_i$. Then the family $(x_i, a_i)_{i\in I}$ in $X \times A$ is not compatible and thus does not admit any gluing at the level u . Suppose that this non-compatible family $(x_i, a_i)_{i\in I}$ lies in $[\varphi]$. Then the $(a_i)_{i\in I}$ lie in $[\exists x\,\varphi]$ and, since $[\exists x\,\varphi]$ is a sheaf, so does the gluing a of that compatible family. But as we have seen, there is no reason for the gluing a to arise from some $(x, a) \in [\varphi](u)$. We can so very well obtain an element $a \in [\exists x\,\varphi](u)$ for which there does not exist an element $x \in X(u)$ such that $(x, a) \in [\varphi](u)$. But by construction, there exists a covering $u = \bigvee u_i$ such that, on each piece of this covering, there exists $x_i \in X_i$ such that $(x_i, a|_{u_i}) \in [\varphi](u_i)$. Thinking of the case of sheaves on a topological space, we end up again with the fact that at the neighborhood of each point, there exists an x such that $\varphi(x, a)$ holds, but this element x can vary from one place to the other one.

4.5 Structure of a topos in its internal language

This section intends to provide some – highly non-exhaustive – examples of basic categorical notions, expressed in the internal language of a topos. Let us recall that given a formula φ on an object A, the subobject of A classified by the truth table of φ (see Definition 4.5) is written $\{a|\varphi(a)\}$.

$\boxed{93}$ **Proposition 4.12** *In a topos $\mathcal{E}$, consider*

- *objects A, B;*
- *morphisms $f, g\colon A \rightrightarrows B$ and $h\colon C \longrightarrow B$;*
- *subobjects $A_1 \rightarrowtail A$, $A_2 \rightarrowtail A$ and $B' \rightarrowtail B$;*
- *variables a, a' of type A, b of type B and c of type C.*

The following properties hold:

1. $f = g$ iff $\models f(a) = g(a)$;
2. *f is a monomorphism* iff $\models \big(f(a) = f(a')\big) \Rightarrow \big(a = a'\big)$;
3. *f is an epimorphism* iff $\models \exists a\, f(a) = b$;
4. $A_1 \cap A_2 = \{a|(a \in A_1) \wedge (a \in A_2)\}$;

5. $A_1 \cup A_2 = \{a | (a \in A_1) \vee (a \in A_2)\}$;

6. $\exists_f A_1 = \big\{ b \big| \exists a \big((a \in A_1) \wedge (b = f(a)) \big) \big\}$;

7. $\forall_f A_1 = \big\{ b \big| \forall a \big((f(a) = b) \Rightarrow (a \in A_1) \big) \big\}$;

8. $f^{-1}(B') = \{a | f(a) \in B'\}$;

9. $\mathsf{Im}\,(f) = \{b | \exists a\, f(a) = b\}$;

10. $\mathsf{Ker}\,(f,g) = \{a | f(a) = g(a)\}$;

11. $f \times_B h = \{(a,c) | f(a) = h(c)\}$ *(pullback of f and h).*

Sketch of the proof Let us prove the first statement. By definition of the morphism $=$, $[f(a) = g(a)]$ is the inverse image of the diagonal of B along (f,g), that is the equalizer $\mathsf{Ker}\,(f,g)$. And of course $\mathsf{Ker}\,(f,g) = A$ is equivalent to $f = g$. □

Much more could be said about constructions in terms of the internal logic of the topos. Let us just mention that, even if a topos is only finitely complete and finitely cocomplete, the internal logic of a topos allows somehow handling "arbitrary internal constructions".

We have already observed that Ω^A should be thought as the "object of subobjects" of A. Thus a subobject $S \rightarrowtail \Omega^A$ can be thought as a family of subobjects of A. The following expressions make sense in the internal logic of the topos, with a a variable of type A an Σ a variable of type Ω^A, and they define subobjects of A:

$$\bigcap S = \big\{ a \big| \forall \Sigma \,(\Sigma \in S \Rightarrow a \in \Sigma) \big\}$$

$$\bigcup S = \big\{ a \big| \exists \Sigma \,(\Sigma \in S \wedge a \in \Sigma) \big\}$$

These subobjects are of course the internal intersection and the internal union of the internal family of internal subobjects.

4.6 Boolean toposes

Let us first observe that

94 **Proposition 4.13** *A Heyting algebra H is a Boolean algebra when $a \vee \neg a = 1$ for every element $a \in H$.*

Sketch of the proof One implication is the content of Proposition 3.25. Conversely, $\neg a = (a \Rightarrow 0)$ is the greatest element such that $a \wedge \neg a = 0$. When moreover $a \vee \neg a = 1$, $\neg a$ becomes the complement of a. □

95 **Definition 4.14** *A topos $\mathcal{E}$ is Boolean when Ω is an internal Boolean algebra (see Theorem 3.29).*

In a Boolean topos, given a formula φ, one has thus always $\models \varphi \vee \neg\varphi$: this is the so called law of the *excluded middle*. On the other hand $\neg\neg\varphi = \varphi$ since $\neg\neg$ is just the double-complement CC in a Boolean algebra.

96 **Proposition 4.15** *In a Boolean topos $\mathcal{E}$, the lattice of subobjects of every object is a Boolean algebra.*

Sketch of the proof For every object A, the internal Boolean algebra structure of Ω induces a Boolean algebra structure on the set $\mathcal{E}(A, \Omega)$ of morphisms, thus on the subobjects of A classified by these. $\square$

97 **Example 4.16** *The topos of sheaves on a complete Boolean algebra is Boolean.*

Sketch of the proof Going back to Example 1.12, each $\Omega(u)$ is a Boolean algebra, where the complement of $v \in\downarrow u$ is $\complement v \wedge u$. $\square$

98 **Example 4.17** *If $\mathcal{E}$ is a Boolean topos and $I \in \mathcal{R}$, then the topos $\mathcal{E}/I$ is Boolean as well..*

Sketch of the proof Let us observe the result in the case $\mathcal{E} = \mathsf{Set}$. The topos Set/I is equivalent to Set^I, as observed in Example3.14. But in Set^I, the topos structure is defined pointwise as in Set, thus is Boolean. $\square$

99 **Example 4.18** *Given a group G, the topos of G-sets is Boolean (see Example 2.6).*

Sketch of the proof As Example 3.5 shows, $\Omega = \{\emptyset, G\}$ is the two-point Heyting algebra, which is thus trivially a Boolean algebra. $\square$

4.7 The axiom of choice

The axiom of choice says that given a family $(A_i)_{i \in I}$ of non-empty sets, it is possible to pick up one element a_i in each A_i. Put $A = \coprod_{i \in I} A_i$ and write $p \colon A \longrightarrow I$ for the projection sending all the elements of A_i on i. Saying that the A_i's are non-empty is saying that p is surjective. Picking up an element a_i in each A_i is choosing a section s of p. Therefore we define:

100 **Definition 4.19** *A topos $\mathcal{E}$ satisfies the* axiom of choice *when every epimorphism admits a section.*

Let us mention, without any proof, that

101 **Theorem 4.20**

1. *A topos satisfying the axiom of choice is Boolean.*
2. *A Grothendieck topos satisfying the axiom of choice in localic.*
3. *A localic topos satisfies the axiom of choice if and only if it is Boolean.*

102 **Example 4.21** *The topos of sheaves on a complete Boolean algebra satisfies the axiom of choice.*

Sketch of the proof By Example 4.16 and Theorem 4.20.3. $\square$

103 **Counterexample 4.22** *Given a non-trivial group G, the topos of G-sets does not satisfy the axiom of choice.*

Sketch of the proof Consider the epimorphism $G \longrightarrow \mathbf{1}$. The unique element of $\mathbf{1}$ is by force stable under the action of every element of G, but no element with that property exists in G. Thus there cannot be any morphism $\mathbf{1} \longrightarrow G$. $\square$

4.8 The axiom of infinity

In order to develop arithmetic, and further analysis in a topos, one needs to construct objects like $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and so on.

104 **Definition 4.23** *A* Natural Number Object *in a topos is a triple* $(\mathbb{N}, 0, s)$

$$\mathbf{1} \xrightarrow{\ 0\ } \mathbb{N} \xrightarrow{\ s\ } \mathbb{N}$$

so that, for every other such triple (X, x, σ), there exists a unique morphism χ making the following diagram commutative:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "0"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, dotted, "\chi"] & \mathbb{N} \arrow[d, dotted, "\chi"] \\
& X \arrow[r, "\sigma"'] & X
\end{tikzcd}$$

The uniqueness condition in Definition 4.23 implies that when it exists, a Natural Number Object is unique up to an isomorphism. In the case of sets, it suffices to take for s the "successor" operation: $s(n) = n + 1$.

105 **Definition 4.24** *A topos $\mathcal{E}$ satisfies the* axiom of infinity *when it contains a Natural Number Object.*

106 **Example 4.25** *Every Grothendieck topos satisfies the axiom of infinity.*

Sketch of the proof Let $(\mathcal{C}, \mathcal{T})$ be a site (see Definition 2.1). In the topos of presheaves on $\mathcal{C}$, the Natural Number Object is defined pointwise as in Set. The sheaf associated with this presheaf (see Theorem 2.15) is the Natural Number Object in the topos of sheaves on $(\mathcal{C}, \mathcal{T})$. $\square$

Of course in a topos with a Natural Number Object, one defines at once constants of type $\mathbb{N}$ by putting

$$0, \quad 1 = s \circ 0, \quad 2 = s \circ 1, \quad 3 = s \circ 2, \quad \text{and so on.}$$

But one should be aware that there are in general other constants of type $\mathbb{N}$. For example in the topos $\mathsf{Set} \times \mathsf{Set}$ (see Example 4.17), the object part of the Natural Number Object is the pair $(\mathbb{N}, \mathbb{N})$ and every pair (n, m) of natural numbers is a constant of type $(\mathbb{N}, \mathbb{N})$.

As a first hint on the way to develop arithmetics in a topos, let us just indicate how one can define the addition on $\mathbb{N}$.

107 **Proposition 4.26** *In a topos $\mathcal{E}$ satisfying the axiom of choice, the object $\mathbb{N}$ can be provided with an addition such that $\models \big(s(n) = n+1\big)$, with n a variable of type $\mathbb{N}$.*

Sketch of the proof Consider the following diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "0"] \arrow[dr, "\iota"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, dotted, "\alpha"] & \mathbb{N} \arrow[d, dotted, "\alpha"] \\
& \mathbb{N}^{\mathbb{N}} \arrow[r, "s^{\mathbb{N}}"'] & \mathbb{N}^{\mathbb{N}}
\end{tikzcd}$$

where ι corresponds to the identity on $\mathbb{N}$ by Cartesian closedness. The definition of a Natural Number Object forces the existence of the morphism α, which corresponds, again by Cartesian closedness, to a morphism $\mathbb{N}\times\mathbb{N}\longrightarrow\mathbb{N}$ which is the expected addition. $\square$

Let us mention another possible approach of the axiom of infinity.

108 **Definition 4.27** *An object of a topos $\mathcal{E}$ is* infinite *when there exists an isomorphism $X\coprod\mathbf{1}\xrightarrow{\cong}X$.*

109 **Proposition 4.28** *A topos $\mathcal{E}$ satisfies the axiom of infinity if an only if it contains an infinite object.*

Sketch of the proof Given a natural number object $(\mathbb{N},0,s)$, the factorization

$$(s,0)\colon \mathbb{N}\coprod\mathbf{1}\longrightarrow\mathbb{N}$$

is an isomorphism, proving that $\mathbb{N}$ is infinite;

Conversely consider an isomorphism $\sigma\colon X\coprod\mathbf{1}\xrightarrow{\cong}X$ and write x for the following constant of type X

$$x\colon \mathbf{1}\longrightarrow X\coprod\mathbf{1}\xrightarrow{\ \sigma\ }X.$$

Using the internal logic of the topos $\mathcal{E}$ and going back to the constructions at the end of Section 4.5, one defines $\mathbb{N}$ as the intersection of all the internal subobjects of X which contain x and are stable under σ. $\square$

Chapter 5

Morphisms of toposes

5.1 Logical morphisms

The most immediate notion of a morphism of toposes is that directly inspired by the Definition 3.1 of an elementary topos.

117 **Definition 5.1** *A* logical morphism $F\colon \mathcal{E} \longrightarrow \mathcal{F}$ *of toposes is a functor which preserves finite limits, the Cartesian closed structure and the subobject classifier.*

As you can expect from Theorem 3.2, even if not obvious to prove:

118 **Proposition 5.2** *A logical morphism of toposes preserves finite colimits.* □

And also:

118a **Proposition 5.3** *A logical morphism of toposes preserves the truth table and the validity of every formula.*

Sketch of the proof This is just straightforward from Definition 5.1 and Proposition 5.2, except for the universal quantifier whose explicit construction is more involved. □

119 **Example 5.4** *Given a morphism $f\colon I \longrightarrow J$ in a topos $\mathcal{E}$, pulling back along f yields a logical morphism $\mathcal{E}/J \longrightarrow \mathcal{E}/I$.*

Sketch of the proof The proof is rather involved, as already that of Proposition 3.13. In the particular case of sets (see Example 3.14), the statement becomes equivalent to proving that the functor

$$\mathsf{Set}^J \longrightarrow \mathsf{Set}^I, \quad (A_j)_{j\in J} \mapsto \left(A_{f(i)}\right)_{i\in I}$$

is a logical morphism. This is trivially the case since the topos structure is defined pointwise in both toposes. □

5.2 Geometric morphisms

Let us now focus on a different notion of morphism of toposes, directly inspired by the case of sheaves on topological spaces. This notion will turn out to be even more useful and important than the notion of logical morphism.

110 **Proposition 5.5** *A continuous mapping $f\colon X \longrightarrow Y$ between topological spaces induces a pair of adjoint functors*

$$f_*\colon \mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(Y), \quad f^*\colon \mathsf{Sh}(Y) \longrightarrow \mathsf{Sh}(X), \quad f^* \dashv f_*$$

with the functor f^* preserving finite limits.

Sketch of the proof With the notation of Section 1.1, given a sheaf F on X, the composite

$$\mathcal{O}(Y) \xrightarrow{f^{-1}} \mathcal{O}(X) \xrightarrow{F} \mathsf{Set}$$

is a sheaf on Y which we define to be $f_*(F)$.

Conversely by Theorem 1.7, a sheaf G on Y corresponds to an étale mapping $g\colon Z \longrightarrow Y$. Pulling g back along f yields an étale mapping over X, whose corresponding sheaf is chosen to be $f^*(G)$. This functor f^* preserves finite limits since so does the pullback operation.

It turns out – even if not obvious – that f^* is left adjoint to f_*. □

Proposition 5.5 can be extended to the more general case of locales. For this observe that:

111 **Lemma 5.6** *A continuous mapping $f\colon X \longrightarrow Y$ between two topological spaces induces a pair of adjoint functors*

$$f_!\colon \mathcal{O}(X) \longrightarrow \mathcal{O}(Y), \quad f^{-1}\colon \mathcal{O}(Y) \longrightarrow \mathcal{O}(X), \quad f^{-1} \dashv f_!$$

with the functor f^{-1} preserving finite meets.

Sketch of the proof The functor f^{-1} preserves finite meets. Since the functor f^{-1} between complete lattices preserves also arbitrary joins, by the adjoint functor theorem, it admits a right adjoint. □

112 **Definition 5.7** *A* morphism *of locales $f\colon L \longrightarrow M$ consists in a pair of adjoint functors*

$$f_!\colon L \longrightarrow M, \quad f^{-1}\colon M \longrightarrow L, \quad f^{-1} \dashv f_!$$

with the functor f^{-1} preserving finite meets.

And as you can expect:

113 **Proposition 5.8** *A morphism of locales $f\colon L \longrightarrow M$ induces a pair of adjoint functors*

$$f_*\colon \mathsf{Sh}(L) \longrightarrow \mathsf{Sh}(M), \quad f^*\colon \mathsf{Sh}(M) \longrightarrow \mathsf{Sh}(L), \quad f^* \dashv f_*$$

with the functor f^* preserving finite limits.

Sketch of the proof Again given a sheaf F on L, the composite

$$M \xrightarrow{f^{-1}} L \xrightarrow{F} \mathsf{Set}$$

is a sheaf on M which we define to be $f_*(F)$.

Since limits of sheaves are computed pointwise, it follows at once that f_* preserves limits. The adjoint functor theorem implies easily the existence of f^*. But proving that f^* preserves finite limits is inferred via an explicit construction, for example in terms of "étale morphisms" of locales. □

The definition in which we are interested is thus the following one:

114 **Definition 5.9** *A geometric morphism $f\colon \mathcal{E} \longrightarrow \mathcal{F}$ of toposes is a pair of adjoint functors*

$$f_*\colon \mathcal{E} \longrightarrow \mathcal{F}, \quad f^*\colon \mathcal{F} \longrightarrow \mathcal{E}, \quad f^* \dashv f_*$$

with the functor f^* preserving finite limits. The functor f_* is called the direct image *functor and the functor f^*, the* inverse image *functor.*

The case of Grothendieck toposes provides at once some other examples:

114c **Example 5.10** *Every site $(\mathcal{C}, \mathcal{T})$ induces a corresponding geometric morphism $\mathsf{Sh}(\mathcal{C}, \mathcal{T}) \longrightarrow \mathsf{Pr}(\mathcal{C})$ between the toposes of sheaves and presheaves.*

Sketch of the proof By Theorem 2.15. □

114a **Proposition 5.11** *Given a Grothendieck topos $\mathcal{E}$, there exists a unique (up to isomorphism) geometric morphism $f\colon \mathcal{E} \longrightarrow \mathsf{Set}$.*

Sketch of the proof An inverse image functor $f^*\colon \mathsf{Set} \longrightarrow \mathcal{E}$ is such that $f^*(\mathbf{1}) = \mathbf{1}$, since it preserves finite limits. It preserves also arbitrary coproducts, since it has a right adjoint f_*. But a set A can be written $A = \coprod_{a \in A}\{a\}$; this forces the definition $f^*(A) = \coprod_{a \in A} \mathbf{1}$ and thus the uniqueness of f^*.

To prove the existence, write $\mathcal{E}$ as the topos of sheaves $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ on a site $(\mathcal{C}, \mathcal{T})$. One gets a geometric morphism when composing the following two geometric morphisms:

$$\mathsf{Sh}(\mathcal{C}, \mathcal{T}) \underset{i}{\overset{a}{\rightleftarrows}} \mathsf{Pr}(\mathcal{C}) \underset{\lim}{\overset{\Delta}{\rightleftarrows}} \mathsf{Set}.$$

i is the inclusion functor and a is its left adjoint, the associated sheaf functor (see Theorem 2.15). Δ is the functor applying a set A on the constant presheaf on A: Δ preserves all limits and colimits since these are computed pointwise in $\mathrm{Pr}(\mathcal{C})$. Moreover Δ has both a left and a right adjoint, namely the functors applying a presheaf F on $\mathsf{colim}\, F$ and $\lim F$. □

115 **Example 5.12** *When X is a Hausdorff topological space, the points of X are in bijection with the geometric morphisms $\mathsf{Set} \longrightarrow \mathsf{Sh}(X)$.*

Sketch of the proof By Proposition 5.5, each continuous mapping $\{x\} \rightarrowtail X$ induces a geometric morphism $\mathsf{Set} \cong \mathsf{Sh}(\{x\}) \longrightarrow \mathsf{Sh}(X)$. Conversely, a geometric morphism $\mathsf{Set} \cong \mathsf{Sh}(\{x\}) \longrightarrow \mathsf{Sh}(X)$ restricts on the locales of subobjects of $\mathbf{1}$ of both toposes, that is, yields a morphism of locales $f\colon \{\emptyset, \{*\}\} \longrightarrow \mathcal{O}(X)$. Since $\mathbf{1}$ and X are Hausdorff spaces (this assumption is even too strong; "sober" suffices), this is the same as a continuous function $\mathbf{1} \longrightarrow X$. □

This justifies the next definition:

116 **Definition 5.13** *A* point *of a Grothendieck topos $\mathcal{E}$ is a geometric morphism* $\mathsf{Set} \longrightarrow \mathcal{E}$.

5.3 Coherent and geometric formulæ

As Proposition 5.3 indicates, a logical morphism of toposes preserves the whole internal logic of the toposes. But what about the geometric morphisms? Given a geometric morphism $f : \mathcal{E} \longrightarrow \mathcal{F}$ of toposes, by adjunction and definition, the direct image functor f_* preserves limits while the inverse image functor f^* preserves colimits and finite limits. So f^* will preserve the part of the internal logic which can be defined in terms of finite limits and colimits.

140 **Definition 5.14** *In a topos $\mathcal{E}$, the* coherent terms *are those obtained as in Definition 4.2, when applying the following restrictions:*

1. *in entries 3, 4, 5, the terms involved are chosen coherent;*
2. *the terms as in entry 6 do not appear.*

The coherent formulæ *are those obtained as in Definition 4.3, when applying the following restrictions:*

1. *the symbols $\in$, $\neg$, $\Rightarrow$ and $\forall$ do not appear;*
2. *in entries 2, 7, the terms involved are chosen coherent;*
3. *in entries 5, 6, 7, φ and ψ are coherent formulæ.*

141 **Proposition 5.15** *Let $f : \mathcal{E} \longrightarrow \mathcal{F}$ be a geometric morphism of toposes. The inverse image functor $f^* : \mathcal{F} \longrightarrow \mathcal{E}$ preserves the truth table of every coherent formula.*

Sketch of the proof Going back to Definitions 4.4 and 4.5, one observes at once that the realization of every coherent term and the truth table of every coherent formula are constructed exclusively in terms of fine limits and colimits, thus are preserved by f^*. $\square$

But as far as the validity of a formula is concerned, one has even more:

142 **Proposition 5.16** *Let $f : \mathcal{E} \longrightarrow \mathcal{F}$ be a geometric morphism of toposes. Given two coherent formulæ φ and ψ, the inverse image functor $f^* : \mathcal{F} \longrightarrow \mathcal{E}$ preserves the validity of φ, $\neg\varphi$ and $\varphi \Rightarrow \psi$.*

Proof By Proposition 5.15, the truth tables of φ and ψ are preserved, thus also their possible validity.

In a Heyting algebra

$$(u \Rightarrow v) = 1 \ \text{ iff } \ 1 \leq (u \rightarrow v) \ \text{ iff } \ 1 \wedge u \leq v \ \text{ iff } \ u \leq v.$$

In particular, $\models (\varphi \Rightarrow \psi)$ is equivalent to $[\varphi] \leq [\psi]$ and this validity is preserved by f^* since it preserves the truth tables of φ and ψ. The case of $\neg\varphi$ is the special case $\psi = \mathsf{false}$. $\square$

142a **Corollary 5.17** *Let $f\colon \mathcal{E} \longrightarrow \mathcal{F}$ be a geometric morphism of toposes. Given a coherent formula φ, the inverse image functor $f^*\colon \mathcal{F} \longrightarrow \mathcal{E}$ preserves the validity of $\exists! x\, \varphi(x)$.*

Sketch of the proof The validity of $\exists! x\, \varphi(x)$ is equivalent to the validity of both formulæ

$$\models \exists x\, \varphi(x), \qquad \models \big(\varphi(y) \wedge \varphi(z)\big) \Rightarrow \big(y = z\big)$$

and one concludes by Proposition 5.16. □

Going back to the observation at the end of Section 4.2, in the case of Grothendieck toposes, the inverse image functor f^* of a geometric morphism preserves arbitrary unions of subobjects, thus will preserve the truth table of an arbitrary disjunction $\bigvee_{i\in I} \varphi_i$ as soon as it preserves the truth table of each individual formula φ_i. Therefore, leaving to the reader the precise formulation of the following definition:

140g **Definition 5.18** *In a Grothendieck topos, geometric terms and geometric formulæ are defined as in Definition 5.14, but allowing arbitrary disjunctions.*

Of course, we get at once:

140gr **Proposition 5.19** *In the case of Grothendieck toposes, the various results of this section carry over to the case of geometric formulæ.* □

5.4 Grothendieck topologies revisited

Our purpose is now to exhibit an important class of geometric morphisms. We borrow our intuition from the case of Grothendieck toposes.

Let $(\mathcal{C}, \mathcal{T})$ be a site. In Proposition 2.18, we have observed that the subobject classifier of the corresponding topos $\mathsf{Pr}(\mathcal{C})$ of presheaves is given by

$$\Omega(C) = \{S | S \text{ is a subobject of } \mathcal{C}(-, C)\}$$

while for a morphism $f \in \mathcal{C}$, $\Omega(f)$ acts by pulling back along f. By condition 3 in Definition 2.1, $\mathcal{T}$ is a subobject of Ω. Let us write $j\colon \Omega \longrightarrow \Omega$ for its characteristic morphism. Thus, given R a subobject of $\mathcal{C}(-, C)$, we have

$$j_C(R)(D) = \{g\colon D \to C | \mathcal{C}(-, g)^{-1}(R) \in \mathcal{T}(D)\}.$$

120 **Proposition 5.20** *The natural transformation j as above makes commutative the following diagrams:*

$$\begin{tikzcd} \Omega \arrow[d, "t"'] \arrow[dr, "t"] & \\ \Omega \arrow[r, "j"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega \arrow[r, "j"] \arrow[dr, "j"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega \times \Omega \arrow[r, "j \times j"] \arrow[d, "\wedge"'] & \Omega \times \Omega \arrow[d, "\wedge"] \\ \Omega \arrow[r, "j"'] & \Omega \end{tikzcd}$$

Sketch of the proof We refer freely to the three conditions in Definition 2.1. The first diagram is commutative because $\mathcal{T}$ is closed under pullbacks. The inclusion $R \subseteq j_C(R)$ is trivial, from which already $j_C(R) \subseteq j_C j_C(R)$; the other inclusion follows from the third axiom for a Grothendieck topology; this takes care of the second diagram. The last diagram commutes because pulling back preserves intersections and each $\mathcal{T}(D)$ is stable under finite intersections. □

121 **Theorem 5.21** *Let $\mathcal{C}$ be a small category. There exists a bijection between*

1. *the Grothendieck topologies $\mathcal{T}$ on $\mathcal{C}$;*
2. *the morphisms $j\colon \Omega \longrightarrow \Omega$, in the topos $\mathsf{Pr}(\mathcal{C})$ of presheaves, which make commutative the three diagrams of Proposition 5.20.*

Sketch of the proof We have just seen how to construct j from $\mathcal{T}$. Conversely given j, it suffices to define $\mathcal{T}$ as the subobject of Ω classified by j. The rest is routine. □

5.5 Internal topologies and sheaves

With in view Theorem 5.21, we define:

122 **Definition 5.22** *A* topology *in a topos $\mathcal{E}$ is a morphism $j\colon \Omega \longrightarrow \Omega$ satisfying*

$$j \circ t = t, \quad j \circ j = j, \quad j \circ \wedge = \wedge \circ (j \times j)$$

(see the diagrams in Proposition 5.20).

As in Definition 2.21, we introduce a corresponding closure operator:

123 **Definition 5.23** *Let $\mathcal{E}$ be a topos an $j\colon \Omega \longrightarrow \Omega$ a topology in $\mathcal{E}$. Given a subobject $S \rightarrowtail P$ with characteristic morphism $\varphi\colon P \longrightarrow \Omega$, the subobject $\overline{S} \rightarrowtail P$ classified by $j \circ \varphi$ is called the* closure *of S.*

124 **Example 5.24** *Consider a site $(\mathcal{C}, \mathcal{T})$ and, by Theorem 5.21, the corresponding topology $j\colon \Omega \longrightarrow \Omega$ in the topos $\mathsf{Pr}(\mathcal{C})$ of presheaves. The closure operators defined respectively in 2.21 and 5.23 coincide.*

Sketch of the proof Just observe the form of $j_C(R)(D)$ given at the beginning of Section 5.4. □

125 **Definition 5.25** *Let $\mathcal{E}$ be a topos and $j\colon \Omega \longrightarrow \Omega$ a topology in $\mathcal{E}$. Given a subobject $S \rightarrowtail P$*

1. *S is* closed *in P when $\overline{S} = S$;*
2. *S is* dense *in P when $\overline{S} = P$.*

126 **Example 5.26** *Consider a site $(\mathcal{C}, \mathcal{T})$ and, by Theorem 5.21, the corresponding topology $j\colon \Omega \longrightarrow \Omega$ in the topos $\mathsf{Pr}(\mathcal{C})$ of presheaves. The dense subobjects of a representable presheaf are exactly its covering sieves.*

Sketch of the proof The closure of a subobject R of $\mathcal{C}$ is the subobject $j_C(R)$ described at the beginning of Section 5.3. If R is dense, choosing $g = \mathsf{id}_C$ in this expression shows that $R \in \mathcal{T}(C)$. The converse holds by stability of $\mathcal{T}$ under pullbacks. □

Definition 2.1 suggests to define further:

127 **Definition 5.27** *Let $\mathcal{E}$ be a topos and $j\colon \Omega \longrightarrow \Omega$ a topology in $\mathcal{E}$. An object $F \in \mathcal{E}$ is called a j-sheaf when it is orthogonal to every j-dense subobject.*

Let us recall that the orthogonality condition means that given a dense subobject $s\colon S \rightarrowtail P$, every morphism $f\colon S \longrightarrow F$ factors uniquely through s.

128 **Example 5.28** *Consider a site $(\mathcal{C}, \mathcal{T})$ and, by Theorem 5.21, the corresponding topology $j\colon \Omega \longrightarrow \Omega$ in the topos $\mathsf{Pr}(\mathcal{C})$ of presheaves. The two notions of sheaf in Definitions 2.2 and 5.27 coincide.*

Sketch of the proof Every j-sheaf is a $\mathcal{T}$-sheaf by Example 5.26. Conversely, given a j-dense subobject $s\colon S \rightarrowtail P$, write P as a colimit of representable functors and, by universality of colimits in $\mathsf{Pr}(\mathcal{C})$ and pullback stability of dense subobjects, write s as a colimit of dense sieves. □

129 **Theorem 5.29** *Let $\mathcal{E}$ be a topos and $j\colon \Omega \longrightarrow \Omega$ a topology in $\mathcal{E}$.*

1. *The full subcategory Sh_j of j-sheaves is a topos.*
2. *The Ω-object of Sh_j is the image Ω_j of j in $\mathcal{E}$.*
3. *The inclusion $i\colon \mathsf{Sh}_j \hookrightarrow \mathcal{E}$ preserves the Cartesian closed structure.*
4. *This inclusion i has a left adjoint preserving finite limits. This adjoint is called the* associated sheaf functor

Sketch of the proof Well ... this is a very deep theorem! See Sections 9.2 and 9.3 of [2] for a proof. □

130 **Corollary 5.30** *Let $\mathcal{E}$ be a topos and $j\colon \Omega \longrightarrow \Omega$ a topology in $\mathcal{E}$. The inclusion $i\colon \mathsf{Sh}_j \hookrightarrow \mathcal{E}$ together with the associated sheaf functor constitute a geometric morphism of toposes.* □

5.6 Back to Boolean toposes

This section points out that every topos contains a Boolean subtopos of sheaves. We take advantage of this section to provide an example of working in the internal logic of a topos.

131 **Proposition 5.31** *In a topos $\mathcal{E}$, the double negation $\neg\neg\colon \Omega \longrightarrow \Omega$ is a topology.*

Sketch of the proof We write down the proof in the internal language of the topos $\mathcal{E}$. First in a Heyting algebra, $\neg 1 = 0$ and $\neg 0 = 1$, thus $\neg\neg 1 = 1$.

By definition, $a \leq \neg b$ if and only if $a \wedge b = 0$. Putting $a = \neg c$, if $b \leq c$, we have $\neg c \wedge b \leq \neg c \wedge c = 0$, thus $\neg c \leq \neg b$. So $\neg$ reverses the ordering and therefore, $\neg\neg$ preserves the ordering.

Second, putting $b = \neg a$ we get $a \leq \neg\neg a$; putting further $a = \neg x$, we obtain $\neg x \leq \neg\neg\neg x$. But since $\neg$ reverses the ordering, from $x \leq \neg\neg x$ we get $\neg\neg\neg x \leq \neg x$. So finally, $\neg\neg\neg = \neg$ and $(\neg\neg) \circ (\neg\neg) = \neg\neg$.

Third, since $\neg\neg$ preserves the ordering, $\neg\neg(x \wedge y) \leq \neg\neg x \wedge \neg\neg y$. Conversely

$$\neg(x \wedge y) \wedge x \wedge y = 0 \;\Rightarrow\; \neg(x \wedge y) \wedge x \leq \neg y = \neg\neg\neg y \;\Rightarrow\; \neg(x \wedge y) \wedge x \wedge \neg\neg y = 0.$$

The same trick can be applied to x, yielding $\neg(x \wedge y) \wedge \neg\neg x \wedge \neg\neg y = 0$ and thus $\neg\neg x \wedge \neg\neg y \leq \neg\neg(x \wedge y)$. $\square$

131a **Lemma 5.32** *Given a topos $\mathcal{E}$, in the internal language of $\mathcal{E}$, the Ω-object of the subtopos $\mathcal{E}_{\neg\neg}$ of sheaves for the double negation topology is*

$$\Omega_{\neg\neg} = \{\omega | \neg\neg\omega = \omega\}$$

(the so-called regular *elements of Ω), with of course ω a variable of type Ω.*

Sketch of the proof We know that $\Omega_{\neg\neg}$ is the image of $\neg\neg$ in $\mathcal{E}$, that is

$$\Omega_{\neg\neg} = \{\neg\neg\omega | \omega \in \Omega\}.$$

Since $\neg\neg(\neg\neg\omega) = \neg\neg\omega$ (see Proposition 5.31), this is trivially equivalent to the definition in the statement. $\square$

132 **Theorem 5.33** *Given a topos $\mathcal{E}$ the subtopos $\mathcal{E}_{\neg\neg}$ of sheaves for the double negation topology is a Boolean topos.*

Sketch of the proof Again we write down the proof in the internal language of the topos $\mathcal{E}$. By the third axiom for a topology, the meet of two elements of $\Omega_{\neg\neg}$ is their meet in Ω. On the other hand it is immediate that the join of two elements in $\Omega_{\neg\neg}$ is, in terms of the operations of Ω, $\neg\neg(\omega \vee \omega')$.

Since $\neg\neg\neg = \neg$ (see the proof of Proposition 5.31), $\omega \in \Omega_{\neg\neg}$ implies $\neg\omega \in \Omega_{\neg\neg}$. Let us prove that $\neg\omega$ is the complement of ω in $\Omega_{\neg\neg}$. We have of course $\omega \wedge \neg\omega = 0$. It remains to prove that $\neg\neg(\omega \vee \neg\omega) = 1$. For that, it suffices to prove that $\neg(\omega \vee \neg\omega) = 0$; this will be the case if every ω' such that $\omega' \wedge (\omega \vee \neg\omega) = 0$ is itself 0. By distributivity we have $(\omega' \wedge \omega) \vee (\omega' \wedge \neg\omega) = 0$, thus both $\omega' \wedge \omega = 0$ and $\omega' \wedge \neg\omega = 0$. This second equality implies $\omega' \leq \neg\neg\omega = \omega$ and putting this in the first equality, we obtain $\omega' = 0$. $\square$

Chapter 6

Classifying toposes

This last chapter intends to show that for a large class of mathematical theories $\mathcal{T}$ (the so-called *coherent theories*), there exists a somehow "generic" model of $\mathcal{T}$ in some Grothendieck topos: a model from which one can recapture all the models of $\mathcal{T}$ in all the possible Grothendiek toposes. The proofs are sometimes technically involved; they can be found in Chapter 4 of [2].

6.1 What is a classifying topos?

It is well-known that given an algebraic theory – that of groups, rings, R-modules, Heyting algebras, and so on – the models of such a theory in a category $\mathcal{C}$ with finite products can equivalently be defined as finite product preserving functors $\mathcal{T} \longrightarrow \mathcal{C}$, where $\mathcal{T}$ is itself a category with finite products. In these cases, $\mathcal{T}$ is the dual of the category of finitely generated free models of $\mathbb{T}$.

Let us begin with – in a sense – a more general definition.

160 **Definition 6.1** *Let $\mathbb{T}$ be a mathematical theory whose models in every Grothendieck topos $\mathcal{F}$ can equivalently be presented as functors $\mathcal{T} \longrightarrow \mathcal{F}$ with adequate properties, on a given small category $\mathcal{T}$. The theory $\mathbb{T}$ admits a classifying topos when there exists a Grothendieck topos $\mathcal{E}[\mathbb{T}]$ and a $\mathbb{T}$-model $M \colon \mathbb{T} \longrightarrow \mathcal{E}[\mathbb{T}]$ such that, for every Grothendieck topos $\mathcal{F}$, the category of $\mathbb{T}$-models in $\mathcal{F}$ is equivalent to the category of geometric morphisms $f \colon \mathcal{F} \longrightarrow \mathcal{E}[\mathbb{T}]$, the equivalence associating with a geometric morphism f the model $f^* \circ M$.*

Of course the theories considered in Definition 6.1 are much more general than just algebraic theories ... but on the other hand the range of categories in which one considers the $\mathbb{T}$-models, and for which the results of this chapter will apply, are only the Grothendieck toposes.

161 **Proposition 6.2** *Let $\mathbb{T}$ be a theory admitting the classifying topos $\mathcal{E}[\mathbb{T}]$. The category of points of $\mathcal{E}$ is equivalent to the category of $\mathbb{T}$-models in* Set.

Sketch of the proof Just by Definitions 5.13 and 6.1. $\square$

6.2 The theory classified by a topos

One often says – even if this sentence does not make any mathematical sense – that a flat functor is one *which preserves all existing and even unexisting finite limits.* A basic example is that of a representable functor

$$\mathcal{C}(C,-)\colon \mathcal{C} \longrightarrow \mathsf{Set}$$

which somehow is "predestinated" to preserve all limits, whatever the category $\mathcal{C}$.[1] And since in Set finite limits commute with filtered colimits, a functor $F\colon \mathcal{C} \longrightarrow \mathsf{Set}$ which is a filtered colimit of representable functors should also be considered as "predestinated" to preserve finite limits.

Let us recall that given a functor $F\colon \mathcal{C} \longrightarrow \mathsf{Set}$, its category $\mathsf{Elt}(F)$ of elements has for objects the pairs (C,x), with $x \in F(C)$; the arrows are the morphisms in $\mathcal{C}$ preserving the corresponding elements. The functor $\mathsf{Elt}(F) \longrightarrow [\mathcal{C},\mathsf{Set}]$ mapping (C,x) on $\mathcal{C}(C,-)$ is contravariant and its colimit is F. For this colimit being filtered, the category $\mathsf{Elt}(F)$ has thus to be cofiltered.

200 **Definition 6.3** *A functor $F\colon \mathcal{C} \longrightarrow \mathsf{Set}$ is* flat *when its category of elements is cofiltered.*

Everything has been done so that:

201a **Proposition 6.4** *A flat functor $F\colon \mathcal{C} \longrightarrow \mathsf{Set}$ preserves all existing finite limits.* □

But more importantly:

201k **Theorem 6.5** *Given a small category $\mathcal{C}$, a functor $F\colon \mathcal{C} \longrightarrow \mathsf{Set}$ is flat if and only if its left Kan extension along the Yoneda embedding preserves finite limits.*

Sketch of the proof We have the situation

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Y"] \arrow[dr, "F"'] & {[\mathcal{C}^{\mathsf{op}},\mathsf{Set}]} \arrow[d, "\mathsf{Lan}_Y F"] \\ & \mathsf{Set} \end{tikzcd} \qquad Y(C) = \mathcal{C}(-,C)$$

where the triangle is commutative (up to an isomorphism), because Y is full and faithful. Moreover

$$\mathsf{Lan}_Y F(G) \cong \mathsf{colim}\,_{(C,x)} G(C)$$

where (C,x) runs through the category of elements of F. The proof reduces to lengthy but routine computations on the Yoneda isomorphism, finite limits and filtered colimits in Set. □

[1] Another class of functors "predestinated" to preserve all limits are those having a left adjoint; they turn out to be flat.

201c **Corollary 6.6** *Let $\mathcal{C}$ be a small category with finite limits. A functor $F\colon \mathcal{C} \longrightarrow \mathsf{Set}$ is flat if and only if it preserves finite limits.*

Sketch of the proof The Yoneda embedding preserves finite limits. $\square$

Some authors use the notation $\mathsf{Lan}_Y F(G) = F \otimes G$: flatness can so be rephrased as $F \otimes -$ preserving finite limits. This throws light on the origin of the terminology *flat*, when thinking of the case of modules on a ring.

We are now ready to define flat functors to Grothendieck toposes.

202 **Definition 6.7** *Let $\mathcal{C}$ be a small category and $\mathcal{E}$, a Grothendieck topos. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{E}$ is flat when its left Kan extension along the Yoneda embedding preserves finite limits.*

162 **Theorem 6.8** *Every Grothendieck topos $\mathcal{E}$ is the classifying topos of a mathematical theory $\mathbb{T}$.*

Sketch of the proof Write $\mathcal{E}$ as the topos of sheaves on a site $(\mathcal{C}, \mathcal{T})$. We shall prove that the topos $\mathcal{E}$ classifies the theory $\mathbb{T}$ of flat functors on $\mathcal{C}$, transforming the covering sieves into colimit cocones. The generic $\mathbb{T}$-model in $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ will be the composite of the Yoneda embedding and the associated sheaf functor

$$\mathcal{C} \xrightarrow{\;Y\;} \mathsf{Pr}(\mathcal{C}) \xrightarrow{\;a\;} \mathsf{Sh}(\mathcal{C}, \mathcal{T}).$$

Let us first observe that $a \circ Y$ is flat. Indeed the left Kan extension of Y along Y is just the identity and since a has a right adjoint, the left Kan extension of $a \circ Y$ along Y is obtained by further composing with a. The Kan extension of $a \circ Y$along Y is thus a, which preserves finite limits (see Theorem 2.15); so $a \circ Y$ is flat. Next given a covering sieve $r\colon R \rightarrowtail \mathcal{C}(-, C)$, R – as every presheaf – is the colimit of the representable functors over it. That colimit is preserved by a and since R is covering, $a(r)$ is an isomorphism. Thus $a \circ Y$ transforms the sieve R into a colimit cocone and $a \circ Y$ is a $\mathbb{T}$-model.

Next, given a geometric morphism $f\colon \mathcal{F} \longrightarrow \mathcal{E}$ between Grothendieck toposes, f^* has a right adjoint f_*. Therefore again, the left Kan extension of $f^* \circ a \circ Y$ along the Yoneda embedding is simply the composite of f^* with the left Kan extension of $a \circ Y$, that is $f^* \circ a$. This composite preserves finite limits and thus $f^* \circ a \circ Y$ is flat. And since f^* preserves all colimits, $f^* \circ a \circ Y$ remains a $\mathbb{T}$-model.

Conversely let $M\colon \mathcal{C} \longrightarrow \mathcal{F}$ be a $\mathbb{T}$-model. The left Kan extension of M along $a \circ Y$ yields a functor $g^*\colon \mathsf{Sh}(\mathcal{C}, \mathcal{T}) \longrightarrow \mathcal{F}$. From the construction of the Kan extension in terms of colimits, one infers that g^* preserves colimits. By the adjoint functor theorem, g^* has a right adjoint g_*. It remains to infer the left exactness of the left Kan extension g^* from the flatness of M. $\square$

164 **Corollary 6.9** *The theory of flat functors on a small category $\mathcal{C}$ admits the topos $\mathrm{Pr}(\mathcal{C})$ of presheaves as classifying topos, with the Yoneda embedding as generic model.*

Sketch of the proof In the proof of Theorem 6.8, simply put $\mathcal{E} = \mathrm{Pr}(\mathcal{C})$, which is the topos of sheaves for the topology on $\mathcal{C}$ whose only covering sieves are the identities. $\square$

Let us make clear that non-isomorphic sites can very well give rise to equivalent toposes of sheaves. Thus a same Grothendieck topos is in general the classifying topos of various mathematical theories, these theories having thus equivalent categories of models in all Grothendieck toposes. This is already the case for the theories of flat functors, since non equivalent categories can very well admit equivalent toposes of presheaves.

6.3 Coherent and geometric theories

Let us treat first the case of a coherent theory, which makes sense in every elementary topos.

150 **Definition 6.10** *In a topos $\mathcal{E}$, by an* operation *of type A with free variables of types $A_1, \ldots, A_n$, we mean a morphism*

$$A_1 \times \cdots \times A_n \longrightarrow A$$

For example a group $(G, +)$ has three operation

$$0\colon \mathbf{1} = G^0 \longrightarrow G, \quad +\colon G \times G \longrightarrow G, \quad -\colon G \longrightarrow G.$$

151 **Definition 6.11** *In a topos $\mathcal{E}$, by a* relation *with free variables of types $A_1, \ldots, A_n$, we mean a subobject*

$$R \rightarrowtail A_1 \times \cdots \times A_n.$$

For example, an ordering on an object A will be a binary relation on A: a subobject $R \rightarrowtail A \times A$.

Leaving to the reader the task of writing down an exhaustive definition in the spirit of those in Section 4.1, we put for short:

152 **Definition 6.12** *A* coherent theory *consists in giving*

- *type symbols;*
- *constants with a prescribed type;*
- *operation symbols $\tau\colon (A_1, \ldots, A_n) \longrightarrow A$ from a finite sequence of types to a single type;*
- *relation symbols R on finite sequences $(A_1, \ldots, A_n)$ of types;*
- *axioms of the form $\models (\varphi \Rightarrow \psi)$ where φ and ψ are coherent formulæ.*

Of course a constant can equivalently be seen as an operation on the empty sequence of types. Let us also recall that $\models \varphi$ is the same as $\models (\mathsf{true} \Rightarrow \varphi)$ and $\models \neg\varphi$ is the same as $\models (\varphi \Rightarrow \mathsf{false})$. Thus Definition 6.12 takes care of these cases.

Again, leaving to the reader the task of writing down a precise definition, we put for short:

153 **Definition 6.13** *Let $\mathcal{T}$ be a coherent theory. A* model *of $\mathcal{T}$ in a topos $\mathcal{E}$ consists first in specifying*

- *an object $\ulcorner A \urcorner$ of $\mathcal{E}$ for each type of the theory;*
- *a morphism $\mathbf{1} \longrightarrow \ulcorner A \urcorner$ of $\mathcal{E}$ for each constant of type A;*
- *a morphism $\ulcorner A_1 \urcorner \times \cdots \times \ulcorner A_n \urcorner \longrightarrow \ulcorner A \urcorner$ for each operation τ as in Definition 6.12;*
- *a subobject $\ulcorner R \urcorner \rightarrowtail \ulcorner A_1 \urcorner \times \cdots \times \ulcorner A_n \urcorner$ in $\mathcal{E}$ for each relation R as in Definition 152.*

Those data constitute a model of $\mathcal{T}$ when all the axioms of $\mathcal{T}$ become valid formulæ in the internal logic of the topos $\mathcal{E}$.
A morphism of such models consists in a family of morphisms of $\mathcal{E}$, one for each type of the theory, in such a way that all the operations and relations of the theory are preserved by this family of morphisms.

And everything has been done so that:

154 **Proposition 6.14** *The inverse image functor of a geometric morphism between elementary toposes preserves the models of every coherent theory.*

Sketch of the proof Just by Propositions 5.15 and 5.16. □

We leave once more to the reader an exhaustive formulation of the following definition.

210 **Definition 6.15** *In the context of Grothendieck toposes, a* geometric theory *is defined analogously to a coherent theory, but allowing φ and ψ in Definition 6.12 to be geometric formulæ(see Defintion 5.18).*

Obviously, in the case of Grothendieck toposes, we get the following generalization of Proposition 6.14.

154g **Proposition 6.16** *The inverse image functor of a geometric morphism between Grothendieck toposes preserves the models of a geometric theory.* □

6.4 The classifying topos of a geometric theory

This last section will show in particular that one can handle the operations and relations of a coherent or geometric theory in terms of cones and cocones as done earlier in this chapter. We shall reduce our attention to the case of Grothendieck toposes and therefore, we shall work at once with the more general notion of geometric theory. In the case of coherent theories, the theory of the classifying topos can be generalized to the case of elementary toposes "over a base topos": in the case of Grothendieck toposes, this base topos is that of sets. But this escapes the scope of these notes.

165 **Lemma 6.17** *Let $\mathcal{C}$ be a small category provided with a set $\mathcal{D}$ of discrete cocones $(f_i\colon C_i \longrightarrow C)_{i\in I}$. Let $\mathbb{T}$ be the theory of flat functors on $\mathcal{C}$ which transform every discrete cocone of $\mathcal{D}$ in an epimorphic family. Then $\mathbb{T}$ has a classifying topos.*

Sketch of the proof Each family $(f_i\colon C_i \longrightarrow C)_{i\in I}$ in $\mathcal{D}$ generates a corresponding sieve $r\colon R \rightarrowtail \mathcal{C}(-,C)$. Since r is a monomorphism, by Proposition 6.4[2], a flat functor to a Grothendieck topos maps r on a monoomorphism and thus the corresponding cocone on a colimit cocone, precisely when it maps it on an epimorphic family. It is easily seen that this property carries over to all the covering sieves constituting the Grothendieck topology $\mathcal{T}$ generated by the sieves R induced from the families in $\mathcal{D}$. The proof of Theorem 6.8 shows that $\mathsf{Sh}(\mathcal{C},\mathcal{T})$ is the expected classifying topos. $\square$

166 **Theorem 6.18** *Every coherent or geometric theory admits a classifying topos.*

Sketch of the proof Let us first construct a graph $\mathcal{G}$ associated with a coherent or geometric theory $\mathbb{T}$. For each finite sequence $(A_1,\ldots,A_n)$ of types, we put a corresponding object in the graph, together with arrows $p_i\colon (A_1,\ldots,A_n)\longrightarrow A_i$ for each index i. For each relation R on the finite sequence $(A_1,\ldots,A_n)$ of types, we put further an object R in $\mathcal{G}$ together with an arrow $r\colon R\longrightarrow(A_1,\ldots,A_n)$. For each operation $\pi\colon (A_1,\ldots,A_n)\longrightarrow A$, we introduce a morphism $\pi\colon (A_1,\ldots,A_n)\longrightarrow A$ in $\mathcal{G}$. Finally, for each constant c of type A, we introduce a morphism $c\colon (\ )\longrightarrow A$, where $(\ )$ indicates the empty sequence of types.

We define $\mathcal{P}$ to be the "path category" of the graph $\mathcal{G}$: the objects are those of $\mathcal{G}$ and the morphisms are the finite sequences of "consecutive" arrows of $\mathcal{G}$. Observe at once that writing $\mathbb{T}_0$ for the theory obtained from $\mathbb{T}$ when omitting all the axioms, a model of $\mathbb{T}_0$ in a Grothendieck topos $\mathcal{E}$ is the same as a functor $F\colon \mathcal{P}\longrightarrow\mathcal{E}$ such that $F(A_1,\ldots,A_,) = F(A_1)\times\cdots\times F(A_n)$ and $F(r)$ is a monomorphism for each relation R.

Adding to $\mathcal{P}$ the required factorizations, one forces $(A_1,\ldots,A_n)$ to become an actual finite product and $(R,\mathsf{id}_R,\mathsf{id}_R)$ to become the pullback of r with itself, that is, we force r to become a monomorphism. Just for the sake of facility, let us further add formally finite limits to the category so obtained.[3] We end up eventually with a small finitely complete category $\mathcal{C}$ such that the left exact functors $\mathcal{C}\longrightarrow\mathcal{E}$ to a Grothendieck topos correspond exactly to the models of $\mathbb{T}_0$.

It remains to take care of the axioms of $\mathbb{T}$. To achieve this, for every geometric formula φ with free variables $a_1,\ldots,a_n$ of types $A_1,\ldots,A_n$, we shall specify a family of morphisms $(f_i\colon B_i\longrightarrow A_1\times\cdots\times A_n)_{i\in I}$ in $\mathcal{C}$ such that for each exact functor $F\colon \mathcal{C}\longrightarrow\mathcal{E}$ to a Grothendieck topos $\mathcal{E}$

$$[\varphi] = \bigcup_{i\in I} \mathsf{Im}\, F(f_i).$$

We shall do this by induction on the complexity of the geometric formula φ. In the case of a coherent family, the set I of indices will be finite.

[2]The morphism r being a monomorphism is equivalent to the pullback of r with itself being given by twice the identity on R.

[3]One can avoid adding all finite limits and work with flat functors instead of left exact ones. But then some arguments below are less evident.

With the formula true we associate the family reduced to the identity on $\mathbf{1}$ and with the formula false, we associate the empty family, viewed as a family with codomain $\mathbf{1}$.

We consider first the formula $\tau = \sigma$, for two geometric terms τ and σ of type A, which we can assume to have the same free variables of types $A_1, \ldots, A_n$. Observe that since $\mathcal{C}$ has finite limits, the restriction on the form of the geometric terms implies that these have already a realization in $\mathcal{C}$ (see Definitions 5.14 and 4.4). Consider the the pullback

$$\begin{tikzcd} B \arrow[r] \arrow[d, tail, "b"'] & A \arrow[d, tail, "\Delta_A"] \\ A_i \times \cdots \times A_n \arrow[r, "(\ulcorner\tau\urcorner, \ulcorner\sigma\urcorner)"'] & A \times A \end{tikzcd}$$

Given a left exact functor $F\colon \mathcal{C} \longrightarrow \mathcal{E}$ to a Grothendieck topos, $[\tau = \sigma]$ is simply $F(B)$, that is, the image of the single morphism $F(b)$.

Next consider two geometric formulæ φ and ψ with the same free variables of types $A_1, \ldots, A_n$. Assume that we have already constructed corresponding respective families

$$(f_i\colon B_i \longrightarrow A_1 \times \cdots \times A_n)_{i \in I}, \quad (g_j\colon C_j \longrightarrow A_1 \times \cdots \times A_n)_{j \in J}$$

as above. Consider this time the pullbacks

$$\begin{tikzcd} Z_{ij} \arrow[r, "c_{ij}"] \arrow[d, "b_{ij}"'] & C_j \arrow[d, "g_j"] \\ B_i \arrow[r, "f_i"'] & A_1 \times \cdots \times A_n \end{tikzcd}$$

in $\mathcal{C}$. Put further $h_{ij} = f_i \circ b_{ij} = g_j \circ c_{ij}$. Given a left exact functor $F\colon \mathcal{C} \longrightarrow \mathcal{E}$ to a Grothendieck topos,

$$[\varphi \wedge \psi] = \bigcup_{i,j} \big(\mathsf{Im}\, F(f_i) \cap \mathsf{Im}\, F(g_j)\big) = \bigcup_{i,j} \mathsf{Im}\, F(h_{ij})$$

Consider now an arbitrary family $(\varphi_i)_{i \in I}$ of formulæ with the same free variables of types $A_1, \ldots, A_n$. Assume that we have already constructed corresponding families

$$(f_{j_i}\colon B_{j_i} \longrightarrow A_1 \times \cdots \times A_n)_{j_i \in J_i}$$

as above. Given a left exact functor $F\colon \mathcal{C} \longrightarrow \mathcal{E}$ to a Grothendieck topos,

$$\left[\bigvee_{i \in I} \varphi_i\right] = \bigcup_{i \in I,\ j_i \in J_i} \mathsf{Im}\, F(f_{j_i}).$$

Finally consider a geometric formula φ with free variables $x, a_1, \ldots, a_n$ of respective types $X, A_1, \ldots, A_n$ and a family $(f_i\colon B_i \longrightarrow X \times A_1 \times \cdots \times A_n)_{i \in I}$ as above. Writing $p\colon X \times A_1 \times \cdots \times A_n \longrightarrow A_1 \times \cdots \times A_n$ for the projection

$$[\exists x\, \varphi] = \bigcup_{i \in I} \mathsf{Im}\, F(p \circ f_i).$$

We are ready to conclude. A left exact functor $F\colon \mathcal{C} \longrightarrow \mathcal{E}$ to a Grothendieck topos determines a $\mathbb{T}$-model when $\models (\varphi \Rightarrow \psi)$ holds for every axiom of the theory $\mathbb{T}$. This is the case precisely when $[\varphi] \subseteq [\psi]$ that is, when the inclusion $[\varphi] \cap [\psi] \subseteq [\psi]$ is also an epimorphism, that is, an isomorphism. Going back to the last diagram above, this is the case when the family of the b_{ij}'s is mapped by F on an epimorphic family. By Lemma 6.17, we get the existence of a corresponding classifying topos. $\square$

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