# Toposes in Como

School: 24-26 June 2018, Conference: 27-29 June 2018

# Contents

# School lectures

## Some glances at topos theory

Francis Borceux
Université Catholique de Louvain

The notion of *sheaf* on a topological space emerged during the period around the second world war, in order to provide an efficient tool to handle local problems. It admits a straightforward generalization to the case of *locales*: lattices which mimic the properties of the lattice of open subsets of a space. But the striking generalization has been that of a sheaf on a site, that is, a sheaf on a small category provided with a so-called *Grothendieck topology*. That notion became essential in algebraic geometry, through the consideration of *schemes*. In the late sixties, F.W. Lawvere introduced *elementary toposes*: categories satisfying axiomatically the properties typical of the categories of sheaves of sets. Each topos provides a model of intuitionistic logic.

This course intends to give a quick overview of some relevant aspects of topos theory, without entering the details of the proofs. It will assume some reasonable familiarity with category theory.

Lecture 1 will introduce the sheaves on a topological space and on a locale and will exhibit some structural properties of the corresponding *localic topos*. We shall provide an application to ring theory.

Lecture 2 will begin with the notion of a *Grothendieck topology* on a small category and the corresponding notion of sheaf. It will investigate some exactness and structural properties of the corresponding *Grothendieck topos*.

In Lecture 3, we shall switch to the axiomatic of those categories called *elementary toposes* and review some of their important properties.

Lecture 4 will throw some light on the *internal logic* of a topos, which is intuitionistic, and the way to use it in order to prove theorems "elementwise".

Lecture 5 will investigate the *morphisms of toposes*, both the *logical* ones and the *geometrical* ones. The link will be made with internal notions of topology and sheaf.

We shall conclude in Chapter 6 with the notion of the *classifying topos* $\mathcal{E}(\mathbb{T})$ of a theory $\mathbb{T}$: a Grothendieck topos which contains a generic model $M$ of $\mathbb{T}$. Generic in the sense that every model of $\mathbb{T}$ in whatever Grothendieck topos $\mathcal{F}$ can be reconstructed from $M$ via the geometric morphism of toposes between $\mathcal{F}$ and $\mathcal{E}(\mathbb{T})$.

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## Grothendieck toposes as 'bridges' between theories

Olivia Caramello
Università degli Studi dell'Insubria - Como

We will explain the sense in which Grothendieck toposes can act as unifying 'bridges' for relating different mathematical theories to each other and studying them from a multiplicity of different points of view. We will first present the general view of toposes as 'bridges' with the resulting techniques, and then discuss a number of selected applications of this methodology in different mathematical fields.

References:

[1] O. Caramello *Grothendieck toposes as unifying 'bridges' in Mathematics*, Mémoire d'habilitation à diriger des recherches, Université de Paris 7, 2016, available at `www.oliviacaramello.com`.

[2] O. Caramello, *Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'*, Oxford University Press, 2017.

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## The power of the simplest arithmetic examples of Grothendieck toposes

Alain Connes
IHÉS

I will describe the joint work with C. Consani in which the simplest arithmetic examples of Grothendieck toposes appear as deeply related to zeta functions and noncommutative geometry.

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## Grothendieck toposes as generalised spaces

Laurent Lafforgue
IHÉS

We will present the basic definitions and results of Grothendieck toposes theory, as it was developed in SGA 4.

First part (two hours):
· Sites and associated toposes
· Giraud's theorem
· Representability critera

Second part (two hours):
· Morphisms of toposes
· Subtoposes
· Canonical and subcanonical topologies

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# Invited conference talks

## Cohomology to logic and back

Tibor Beke

Department of Mathematics, University of Massachusetts

It is a running theme of this conference that toposes come in many guises. Categories of geometric objects such as topological spaces, simplicial sets, varieties and schemes allow well-behaved embeddings into the category of toposes and geometric morphisms, just as categories of coherent theories, models of set theories or definable sets do. This sometimes allows one to 'borrow' an idea from one context, express it in the generality of toposes, and apply it in a context that would seem completely alien at first.

The notion of cohomology originated as an algebraic dual to the geometric notion of (simplicial) homology. It is quite a miracle that a well-behaved notion of cohomology (but not homology!) can be defined for an arbitrary topos. This, in turn, allows one to ask questions about logical notions associated to cohomology, or cohomology groups of logical theories.

The talk will walk through this circle of ideas, using the very simple (and very classical) notion of torsor as starting point. The unifying thread of the discussion is Joyal's notion of 'natural homotopy' or, more or less equivalently, defining invariants of a theory from the set of connected components of its category of models in a topos. If time permits, I hope to talk about cohomology operations and open questions about their 'combinatorial' interpretation.

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## How topos theory can help commutative algebra

Ingo Blechschmidt

University of Augsburg / Max Planck Institute for Mathematics in the Sciences, Leipzig

Topos theory and commutative algebra are closely linked: In view of the mantra "toposes are rings", commutative algebra has been informing topos theory, and by employing the internal language of toposes, topos theory has been used to transfer results from commutative algebra to subjects such as algebraic geometry and differential geometry. In this talk, we explore a third link: How topos theory can help commutative algebra and neighboring disciplines.

One such relation is given by new reduction techniques. For instance, there is a way how we can assume, without loss of generality, that any reduced ring is a field.

This technique allows to give a short and simple proof of Grothendieck's generic freeness lemma, a basic theorem used in the setup of the theory of moduli spaces, which substantially improves on the previous somewhat convoluted proofs.

These topos-theoretic reduction techniques cannot generally be mimicked by traditional commutative algebra, and in the special cases where they can, they improve on the traditional methods by yielding fully constructive proofs. The precise sense in which all this is true will be carefully explained in the talk.

A further relation is given by synthetic approaches to algebraic geometry, allowing to treat schemes with all their complex algebro-geometrical structure as plain sets and morphisms between schemes as maps between these sets. Fundamental to this account is the notion of "synthetic quasicoherence", which doesn't have a counterpart in synthetic differential geometry and which endows the relevant internal universes with a distinctive algebraic flavor.

Somewhat surprisingly, the work on synthetic algebraic geometry is related to an age-old question in the study of classifying toposes. We hope to report on recent results by Matthias Hutzler in this regard, and close with an invitation to the many open problems of the field. The talk will begin with an introduction to the internal language of toposes, so as to be accessible to audience members who are not familiar with it and to provide value outside of commutative algebra.

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## On the relation between Grothendieck-Galois and Deligne-Tannaka theories

Eduardo J. Dubuc

University of Buenos Aires

Strong similarities have been long observed between different versions of Galois and Tannaka representation theories. However, these similarities are just of form, and don't allow to transfer any result from one theory to another, in particular Galois theory and Tannaka theory (over vector spaces) remain independent. As a preamble, we will briefly recall the Grothendieck-Galois and the neutral Deligne-Tannaka versions of these theories.

Observing that the category of relations of a Grothendieck topos is a category enriched over sup-lattices, we relate the Galois context to the Tannakian context over the tensor category $s\ell$ of sup-lattices. We develop the case of the (localic) group $G$ of automorphisms of a $Set$-valued fiber functor $F$, on the Galois side, and the Hopf algebra $H$ of endomorphism of a $s\ell$-valued fiber functor $T$, on the Tannaka side. This correspondence is obtained via the category of relations functor $Rel$. We establish an isomorphism between $G$ and $H$ for $T = Rel(F)$, and between the categories of actions of $G$ on a set and the category of comodules of $H$ on a free sup-lattice. This yields an equivalence between the respective recognition theorems. The general case deals with localic groupoids and Hopf algebroids, and we obtain similar results. This concerns the Joyal-Tierney generalisation of Galois theory, and it becomes necessary to work over an arbitrary base topos because change of base techniques become essential and unavoidable. In this talk we will concentrate on the neutral case which is within the scope of a larger audience, and only mention the general case.

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## Isotropy of Toposes

Pieter Hofstra

University of Ottawa

Just as every Grothendieck topos contains a canonical locale, namely its subobject classifier, it also contains a canonical group object, called its isotropy group. This algebraic invariant has many interesting interpretations and connections with other areas. For example, from a logical point of view it encodes certain definable automorphisms. From an algebraic point of view, it embodies a notion of inner, or central, automorphism. It is also closely related to the classical notion of crossed module. And from a computer science perspective, it encodes invertible polymorphic operations. The isotropy group acts canonically on every object of the topos, so we may form the topos of objects for which this action is trivial. Surprisingly, the resulting quotient topos may itself have non-trivial isotropy, giving a possibly transfinite sequence of toposes, much in the same vein as the central series of a group. This phenonemon gives rise to a new ordinal invariant of toposes. In this talk I will give an introductory overview of the isotropy group, its interpretations and applications, and the phenomenon of higher isotropy.

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## Elementary toposes from a predicative point of view

Maria Emilia Maietti

Università degli Studi di Padova

The internal language of an elementary topos is an higher order logic which is impredicative in the sense of Russell-Poincarè.

Inspired by the categorical semantics of the internal language of toposes in [Mai05] in terms of Benabou's fibred categories, we propose a *generalization* of the notion of *elementary topos* whose internal language is *predicative* in the sense of P. Martin-Löf.

Our predicative notion of elementary topos can be viewed as a special case of a further generalization, called "minimalist", that amounts to be a predicative generalization of the notion of quasi-topos when the whole structure is equipped with a natural numbers object. Its internal language is then provided by the extensional level of the Minimalist Foundation in [Mai09].

To produce predicative examples of our generalizations of toposes, as that in [MM], we employ the notion of elementary quotient completion introduced in [MR13] as a generalization of the construction of exact completion on a weakly lex category relative to a Lawvere's elementary doctrine.

Further examples of these structures can be built by using predicative type theories developed by P. Martin-Löf and more recently by V. Voevodsky.

References:

[Mai05] M.E. Maietti. Modular correspondence between dependent type theories and categories including pretopoi e topoi. *Mathematical Structures in Computer Science*, 15(6):1089–1149, 2005.

[Mai09] M.E. Maietti. A minimalist two-level foundation for constructive mathematics. *Ann. Pure Appl. Logic*, 160(3):319–354, 2009.

[MM] M.E. Maietti and S. Maschio. A predicative variant of Hyland's Effective Topos. Submitted.

[MR13] M.E. Maietti and G. Rosolini. Elementary quotient completion. *Theory Appl. Categ.*, 27:445–463, 2013.

[MS05] M.E. Maietti and G. Sambin. Toward a minimalist foundation for constructive mathematics. In L. Crosilla and P. Schuster, editors, *From Sets and Types to Topology and Analysis: Practicable Foundations for Constructive Mathematics*, number 48, pages 91–114, Oxford University Press, 2005.

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## Models of Univalence in Toposes

Andrew Pitts

University of Cambridge

Homotopy Type Theory (HoTT) has re-invigorated research into the theory and applications of the intensional version of Martin-Löf type theory. On the one hand, the language of type theory helps to express synthetic constructions and arguments in homotopy theory and higher-dimensional category theory. On the other hand, geometric and algebraic insights help shed new light on logical and type-theoretic notions. In particular, HoTT takes a path-oriented view of intensional (proof-relevant) equality: proofs of equality of two elements of a type $x, y : A$, i.e. elements of a Martin-Löf identity type $Id_A(x, y)$, behave analogously to paths between two points $x, y$ in a space $A$. The complicated internal structure of intensional identity types relates to the homotopy classes of path spaces. To make this analogy precise and to exploit it, it helps to have a wide range of models of intensional type theory that embody this path-oriented view of equality. In this talk I focus on models in which "path" means an arbitrary function from an interval-like object $I$ in some topos and address the question: what properties of the topos and of $I$ are needed to construct a universe of types, with identity types given by paths out of $I$, satisfying Voevodsky's univalence axiom?

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## Nonconnective structures

Mauro Porta

Université de Strasbourg

It is known since the work of Lawvere that we can represent many algebraic structures in topoi as product preserving functors out of specific categories. For example, if $T_{Ab}$ denotes the opposite of the category of free abelian groups of finite rank, then abelian group objects in a topos $X$ can be represented as $Fun^{\times}(T_{Ab}, X)$. When we replace $X$ with the $\infty$-topos of spaces $\mathcal{S}$ the situation becomes somehow more complicated. In the specific case of $T_{Ab}$ we can use results of D. Quillen and J. Bergner to prove that the $\infty$-category $Fun^{\times}(T_{Ab}, \mathcal{S})$ is modeled by simplicial abelian groups. However, simplicial abelian groups only represent the connective part of the theory of unbounded chain complexes (or of spectra). In this talk I will explain how to adapt the Lawvere theoretical approach to algebraic structures in order to encode non-connective phenomena. As an application, I will sketch a proof of an analytic version of the Hochschild-Kostant-Rosenberg.

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## Homotopy type theory, synthetic topology and probabilistic programming

Bas Spitters

Aarhus University

We will model probabilistic programming in homotopy type theory using the univalence axiom. This is based on the use of homotopy type theory as an 'internal language' for higher toposes. We will add axioms to homotopy type theory which are justified in certain sheaf and realizability models. This allows us to use synthetic topology to present a theory which also includes non-discrete ('continuous') data types, like $[0, 1]$. We will explain how this work relates to the easycrypt proof assistant for computer aided proofs in cryptography and to the augur/v2 programming language for machine learning.

This lecture is based on:


- Florian Faissole and Bas Spitters, Synthetic topology in Homotopy Type Theory for probabilistic programming, preprint 2018, `http://www.cs.au.dk/~spitters/ProbProg.pdf`
- Daniel Huang, Greg Morrisett, and Bas Spitters, An Application of Computable Distributions to the Semantics of Probabilistic Programs, forthcoming


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# A topos-theoretic approach to systems and behavior

David I. Spivak (joint work with Patrick Schultz)
Massachusetts Institute of Technology

A well-functioning logic of behavior is indispensible for studying interactions among real-world systems and processes, for which typical mathematical modeling frameworks range from that of dynamical systems to that of Turing machines. It appears that any behavior—e.g. a trajectory of an ordinary differential equation or an execution of a computer program—takes place over a duration of "time". Thus in [1] we propose the theory of real durations, i.e. translation-invariant intervals, as a formal system for studying behavior.

In this talk, I will recall the topological space $\mathbb{IR}$, called the *interval domain* [2], and discuss how its sheaves—objects in $\mathsf{Shv}(\mathbb{IR})$—can be conceptualized as types of behavior in the context of time. There is a quotient topos $\mathsf{Shv}(\mathbb{IR}) \to \mathsf{Shv}(\mathbb{IR}_{/\rhd})$, defined in terms of the translation action $\rhd$ on $\mathbb{IR}$ by the group $\mathbb{R}$ of real numbers, and we refer to $\mathcal{B} := \mathsf{Shv}(\mathbb{IR}_{/\rhd})$ as the topos of *behavior types.* Using a special behavior type, which we denote $\mathtt{Time} \in \mathcal{B}$, one may recover $\mathsf{Shv}(\mathbb{IR}) \cong \mathcal{B}/\mathtt{Time}$ as a slice topos of $\mathcal{B}$.

I will explain the relationship between $\mathcal{B}$ and the topos studied by Lawvere in [4] and others (e.g. [5]), also in the context of abstract dynamical systems. I will also briefly discuss how to use the internal language of the topos $\mathcal{B}$ to describe the behaviors of both ODEs and state machines, in order to indicate how these two typical—but very different—behavior-modeling frameworks can interact within a single logical formalism, which we call *higher order temporal logic.*

References:

[1] Schultz, P. and Spivak, D.I., 2017. *Temporal Type Theory: A topos-theoretic approach to systems and behavior.* Birkaüser Mathematics (*to appear*).

[2] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M. and Scott, D.S., 2003. Continuous lattices and domains (Vol. 93). Cambridge university press.

[3] Johnstone, P. and Joyal, A., 1982. Continuous categories and exponentiable toposes. *Journal of Pure and Applied Algebra*, 25(3), pp.255-296.

[4] Lawvere, F.W., 1986. State categories and response functors. *preprint.*

[5] Spivak, D.I., Vasilakopoulou, C. and Schultz, P., 2016. Dynamical systems and sheaves. *preprint* arXiv:1609.08086.

# Contributed conference talks

## Topology of interactivity

Stéphane Dugowson
Quartz / Supmca Paris

In a general systemic theory of interactivity that I have been developing for a few years [4], the interaction within a *dynamical family* is defined as a *multiple relation* between *realizations* and *parameters* of the concerned open dynamics. By the way, a *connectivity structure* [1, 2] is naturally associated with every multiple relation [3]. In general, this connectivity structure is not the one of a topological space. However, it gives rise to a localic topos and therefore, at least in the finite case, to a topological space [5] thus giving us a "topology of the interaction". Among the points of this space, there are not only the dynamics of the considered dynamical family, but also new points which represent some *irreducible* sub-families. The purpose of this short presentation is to provide an intuitive overview of these ideas.

References:

[1] Stéphane Dugowson. On connectivity spaces. *Cahiers de Topologie et Gomtrie Diffrentielle Catgoriques*, LI(4):282–315, 2010. http://hal.archives-ouvertes.fr/hal-00446998/fr.

[2] Stéphane Dugowson. Espaces connectifs : reprsentations, feuilletages, ordres, diffologies. *Cahiers de topologie et gomtrie diffrentielle catgorique*, LIV, 2013. `https://hal.archives-ouvertes.fr/hal-01386249`.

[3] Stéphane Dugowson. Structure connective des relations multiples, 5 mai 2015. `https://hal.archives-ouvertes.fr/hal-01150262`.

[4] Stéphane Dugowson. Dynamiques en interaction : une introduction la thorie des dynamiques sous-fonctorielles ouvertes (hal-01357009), 31 aot 2016. `https://hal.archives-ouvertes.fr/hal-01357009`.

[5] Stéphane Dugowson. Toposes of connectivity spaces. Morita equivalences with topological spaces and partially ordered sets in the finite case., 5 mars 2018. `https://hal.archives-ouvertes.fr/hal-01722695`.

## Infinitary generalizations of Deligne's completeness theorem

Christian Espindola
Stockholm University

Given a regular cardinal $\kappa$ such that $\kappa^{<\kappa} = \kappa$ (or any regular $\kappa$ if the Generalized Continuum Hypothesis holds), we study a class of toposes with enough points, the $\kappa$-separable toposes. These are equivalent to sheaf toposes over a site with $\kappa$-small limits that has at most $\kappa$ many objects and morphisms, the (basis for the) topology being generated by at most $\kappa$ many covering families, and that satisfy a further exactness property $T$. We prove that these toposes have enough $\kappa$-points, that is, points whose inverse image preserve all $\kappa$-small limits. This generalizes the separable toposes of Makkai and Reyes, that are a particular case when $\kappa = \omega$, when property $T$ is trivially satisfied. This result is essentially a completeness theorem for a certain infinitary logic that we call $\kappa$-geometric, where conjunctions of less than $\kappa$ formulas and existential quantification on less than $\kappa$ many variables is allowed. For example, the theory of well-orderings is $\kappa$-geometric for $\kappa > \omega$, but it is not geometric and, in fact, cannot be expressed in any finite-quantifier language $\mathcal{L}_{\kappa,\omega}$. We prove that $\kappa$-geometric theories have a $\kappa$-classifying topos having property $T$, the universal property being that models of the theory in a Grothendieck topos with property $T$ correspond to $\kappa$-geometric morphisms (geometric morphisms the inverse image of which preserves all $\kappa$-small limits) into that topos. Moreover, we prove that $\kappa$-separable toposes occur as the $\kappa$-classifying toposes of $\kappa$-geometric theories of at most $\kappa$ many axioms in canonical form, and that every such $\kappa$-classifying topos is $\kappa$-separable. Finally, we consider the case when $\kappa$ is weakly compact and study the $\kappa$-classifying topos of a $\kappa$-coherent theory (with at most $\kappa$ many axioms), that is, a theory where only disjunction of less than $\kappa$ formulas are allowed, obtaining a version of Deligne's theorem for $\kappa$-coherent toposes from which we can derive, among other things, Karp's completeness theorem for infinitary classical logic.

References:

[1] Dickmann, Max: *Large infinitary languages* - North-Holland Publishing Company (1975).

[2] Espíndola, Christian: *Infinitary first-order categorical logic* - To appear in the Annals of Pure and Applied Logic (2017).

[3] Johnstone, Peter: *Sketches of an elephant* - Oxford University Press, New York (2002).

[4] Karp, Carol. *Languages with expressions of infinite length* - North-Holland Publishing Company (1964).

[5] Makkai, Michael - Reyes, Gonzalo: *First-order categorical logic* - Springer (1977).

[6] Makkai, Michael: *A theorem on Barr-exact categories, with an infinitary generalization* - Annals of Pure and Applied Logic 47, no. 3, pp. 225-268 (1990).

# Fibrations of contexts and fibrations of toposes

Sina Hazratpour

School of Computer Science, University of Birmingham

The notions of (op)fibration in the 2-category of toposes and geometric morphisms have close connections to topological properties. For example, every local homeomorphism is an opfibration. This connection is in line with the conception of toposes as generalized spaces.

To study fibrations of toposes, Peter Johnstone has defined fibrations internal to 2-categories in [3]. Johnstone's definition requires neither strictness of 2-categories nor the existence of the structure of strict pullbacks and comma objects in them. Indeed, this definition is particularly well-suited for the class of 2-categories such as 2-category of bounded toposes in which we do not expect diagrams of 1-cells to commute strictly. Moreover, Johnstone also established the equivalence of his definition with the representable definition of internal fibrations. However, this definition is rather complicated and difficult to work with in practice.

I will introduce the 2-category $\mathfrak{Con}$ of contexts developed in [2], and [4]. Among other things $\mathfrak{Con}$ gives a syntactic presentation of finitary fragment of theory of toposes. It also provides us with good handling of strictness.

Borrowing from work of Ross Street in [1], we introduce a syntactic notion of (op)fibration in $\mathfrak{Con}$ which is based on Chevalley's internal characterization of fibrations obtained as a theorem in there.

Using the machinery of classifying toposes, I shall describe how our finitary syntactic definition of (op)fibrations of contexts will give rise to (op)fibrations of toposes in the sense of Johnstone.

References:

[1] R. Street, "Fibrations and Yoneda's lemma in a 2-category", *Lecture Notes in Math., Springer, Berlin*, vol. 420, 1974, pp. 104-133.

[2] S. Vickers, "Sketches for arithmetic universes", 2016,
URL: `https://arxiv.org/abs/1608.01559`

[3] P. Johnstone, "Fibrations and partial products in a 2-category", *Applied Categorical Structures*, vol. 1, 1993, pp. 141–179.

[4] Vickers, S. "Arithmetic universes and classifying toposes", *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 58(4):213–248, 2017.

# Stochastic manifolds

Anatole Khelif

Université de Paris 7

"To do a geometry you do not need a space, you only need an algebra of functions on this would-be space."

A. Grothendieck

The Malliavin Calculus can be seen as a differential calculus on Wiener spaces. It is then possible to establish a new dimensionless differential geometry, for which the Malliavin calculus plays the same role as the one played by the classical differential calculus in the theory of $n$-dimensional manifolds.

Moreover, is it also possible to obtain a Variational Calculus on a random structure, which is built by constraints subjected to infinitesimal variation? This sort of problem is recurrent in Physics and Econometry.

Such a Variational Calculus imposes a reasonnable space of "measurable and regular" functions, with a compatibility between associated differentiation and integration processes, from which a generalized divergence operator.

As such a problem requires an infinite dimensional space, it becomes needed to have an infinite dimensional differential calculus with a related good notion of a divergence.

The Malliavin Calculus provides such a tool. More precisely: in $\mathbb{R}^n$, there is compatibility between differentiation and integration because the Lebesgue measure is translation invariant. Unfortunately, in the case of an infinite dimensional topological vector space $E$, such a non-trivial translation measure does not exist. But there can be quasi-invariant measures $\mu$, that is : there is a dense subspace $H$ of $E$, such that the image measure of $\mu$ by a translation with the vector $h \in H$, admits a density relatively to $\mu$.

A natural is $E$=Wiener space $\mathcal{W}$, with $\mu$=Gaussian measure, and $H$ being the Cameron-Martin space which then is an Hilbert space.

More precisely, given a basis manifold $V$ and a fiber space $F$ on $V$, it is possible to endow the space of the random sections of $F$, witch a reasonable measure so that there is a Variational calculus.

Two particular cases which are extreme case have already been studied : random Brownian fields (maps from $V$ in a Gaussian space), and the set of continuous paths in a Compact Riemannian manifold.

To build the general theory of $\mathbb{D}^\infty$-stochastic manifolds, a source will be the Grothendieck identification of an $n$-dimensional manifold with a sheaf of $C^\infty$-functions; here, $C^\infty$ will be replaced by $\mathbb{D}^\infty(\Omega)$, and a diffeomorphism will be a map between two Gaussian spaces that will keep the $\mathbb{D}^\infty$ property through right-composition and this diffeomorphism will have a canonical "cotangent" linear map.

# A geometric theory for o-minimal structures

Henri Lombardi

Université de Franche-Comté

We work in a pure constructive context, minimalist, à la Bishop.

Constructive real algebra is not well understood. In fact constructive real algebra is fairly distinct from the usual theory of discrete real closed fields, where a sign test is assumed to be given for the order structure.

Most theorems of discrete real algebra fail from a constructive point of view because there is no sign test for constructive real numbers.

Searching to describe the purely algebraic properties of a constructive real number system (e.g. the one of Bishop), we consider that a good way is to construct a geometric theory, as complete as possible, whose a model should be any of the proposed constructive real number systems in the literature.

For the external language, e.g. when we speak about models, we want also to use some weak form of constructive set theory, with intuitionist logic.

We also need a very uniform theory, avoiding properties that can be proven only with the use of dependant choice (as somme results in Bishop's book).

A first attempt, when searching for a coherent theory, leads us to the theory of local real closed rings. The theory of real closed rings can be given in a purely equational form. And the fact that a real number field is a local ring is a way to formulate the dichotomy for overlapping intervals: if $x + y$ is $> 0$ then $x > 0$ or $y > 0$ (with an explicit "or").

But the theory seems still unsatisfactory and incomplete. We think it is necessary to add new sorts in order to describe in a more uniform way the properties of semialgebraic continuous functions on a compact cube. We see in this new context that some infinitary axioms appear to be important and necessary. So we pass from a coherent theory to a more general geometric theory.

Finally we hope to obtain a good geometric theory for o-minimal structures. This is a confirmation of our moto: constructive real is noting else than the first example of (wonderful) constructive o-minimal structures.

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A first draft can be found at
`http://hlombardi.free.fr/Reels-geometriques.pdf`.

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## Grothendieck categories as a bilocalization of linear sites

Julia Ramos González
Universiteit Antwerpen

Grothendieck categories play an essential role in algebraic geometry since the Grothendieck school, and are center stage in non-commutative algebraic geometry since the work of Artin, Stafford, Van den Bergh and others. By the Gabriel-Popescu Theorem, we know that they can be viewed as "linear topoi", that is, as categories of sheaves on linear sites.

The aim of this talk is to describe the relation between Grothendieck categories and linear sites on a bicategorical level.

Fixed a commutative ring $k$, this will be done from two different perspectives:

- On the one hand, inspired by our previous results in [1], we consider:
  - the 2-category $\mathsf{Grt}_k$ of Grothendieck $k$-linear categories with colimit preserving $k$-linear functors and $k$-linear natural transformations;
  - the 2-category $\mathsf{Site}_{k,\mathsf{cont}}$ of $k$-linear sites with continuous $k$-linear functors and $k$-linear natural transformations.
- On the other hand, inspired by the classical setup of topos theory, we consider:
  - the 2-category $\mathsf{Topoi}_k$ of Grothendieck $k$-linear categories with $k$-linear geometric morphisms and $k$-linear morphisms between them;
  - the 2-category $\mathsf{Site}_k$ of $k$-linear sites with $k$-linear morphisms of sites and $k$-linear natural transformations.

From the Gabriel-Popescu theorem, it follows that every Grothendieck $k$-linear category can be realised as a category of sheaves on a $k$-linear site. Moreover, we show that every colimit preserving $k$-linear functor (resp. every $k$-linear opposite geometric morphism) between two categories of sheaves can be obtained as being induced from a "roof" of continuous $k$-linear functors between sites (resp. of $k$-linear morphisms of sites), where the "reversed arrows" are a special type of morphisms of sites called *LC morphisms* (where LC stands for "Lemme de comparaison").

These observations make it natural to view $\mathsf{Grt}_k$ (resp. the *conjugate-opposite bicategory* of $\mathsf{Topoi}_k$, i.e. the bicategory obtained by reversing the direction of both 1-morphisms and 2-morphisms in $\mathsf{Topoi}_k$) as a localization of $\mathsf{Site}_{k,\mathsf{cont}}$ (resp. of $\mathsf{Site}_k$). We make this idea precise by using the localization of bicategories with respect to a class of 1-morphisms developed by Pronk in [2] and further analysed by Tommasini in the series of papers [3, 4, 5].

In addition, we show how this result can be potentially used to make the tensor product of Grothendieck categories from [1] into a bi-monoidal structure on $\mathsf{Grt}_k$.

References:

[1] W. Lowen, J. Ramos González and B. Shoikhet, *On the tensor product of linear sites and Grothendieck categories*, Int. Math. Res. Not. IMRN, 2017, `https://doi.org/10.1093/imrn/rnx072`.

[2] D. Pronk, *Etendues and stacks as bicategories of fractions*, Compositio Math. **102** (1996), no. 3, 24–303.

[3] M. Tommasini, *Some insights on bicategories of fractions: representations and compositions of 2-morphisms* Theory Appl. Categ. **31** (2016), Paper No. 10, 257–329.

[4] ———, *Some insights on bicategories of fractions II - Right saturations and induced pseudofunctors between bicategories of fractions*, `arXiv:1410.5075 [math.CT]`.

[5] ———, *Some insights on bicategories of fractions III - Equivalences of bicategories of fractions*, `arXiv:1410.6395 [math.CT]`.

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## An arithmetic site for $\mathbb{Q}(\sqrt{2})$

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Adele class spaces for number fields are, as A. Connes underlined it in 1995, very important for the spectral realization of Hecke $L$-functions (and so in particular the Riemann zeta function or the Dedekind zeta function of a number field). The semi-ringed topos $(\widehat{\mathbb{N}^{\star}}, (\mathbb{Z} \cup \{-\infty\}, \max, +))$ called by A. Connes and C. Consani the arithmetic site and introduced by them in 2014 provides a algebraic-geometric background for the adèle class space of $\mathbb{Q}$. Thanks to this algebraic-geometric backbground, one could hope in the long term to transfer and adapt to the context of number fields (and in A. Connes' and C. Consani's case $\mathbb{Q}$) ideas coming from Weil's proof of the analogue of the Riemann hypothesis in the function field case.

In my PhD thesis, I introduced for the number field $\mathbb{Q}(\imath)$ a semi-ringed topos similar to the one introduced by A. Connes and C. Consani and which is linked to the adele class space of $\mathbb{Q}(\imath)$ and consequently linked to the Dedekind zeta function of $\mathbb{Q}(\imath)$.

In this short communication, based on ideas of my PhD thesis, on remarks of A. Connes and C. Consani on my PhD thesis and on Shintani's unit theorem, I propose to introduce an arithmetic site of Connes-Consani type for the number field $\mathbb{Q}(\sqrt{2})$.

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## Grothendieck toposes constructively, via arithmetic universes

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Giraud's categorical characterization of Grothendieck toposes requires all set-indexed colimits, and so presupposes an understanding of "set" that is commonly taken to be classical. Elementary topos theory does provide a constructive approach, by defining Grothendieck toposes relative to a base $\mathcal{S}$, an arbitrarily chosen elementary topos: they are the bounded geometric morphisms into $\mathcal{S}$. However, this creates a new problem in that the classifying topos $\mathcal{S}[\mathbb{T}]$ for a geometric theory $\mathbb{T}$ is not a single category, but a family indexed by $\mathcal{S}$. I shall outline how this indexation can be treated in a fibrational way, and how base-free arguments in a single 2-category using the logic of arithmetic universes can be used to prove fibred results for the toposes.

Based on:

"Sketches for arithmetic universes" - arxiv:1608.01559;

"Arithmetic universes and classifying toposes" - *Cahiers de top. geom. diff. cat.* 58(4).

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