Infinitary generalizations of Deligne's completeness theorem

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June 29th, 2018

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Essentially equivalent to Gödel completeness theorem for first-order classical logic.

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GOAL: Replace ω by any cardinal κ with $\kappa^{<\kappa} = \kappa$, obtaining thereby a completeness theorem for infinitary classical logic over $\mathcal{L}_{\kappa^+,\kappa}$, and moreover, for an intuitionistic fragment that we call κ-geometric.

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López-Escobar: the theory of well-orderings is not axiomatizable in $\mathcal{L}_{\kappa,\omega}$ for any κ.

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A κ-separable topos is a topos of sheaves for which there is a site on a category with κ-small limits, at most κ many objects and morphisms, where the basis for the Grothendieck topology is generated by at most κ-many covering families and satisfying a further exactness property T.

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Theorem

(E.) κ-separable toposes have enough κ-points.

The exactness property T

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This leads us to a strong distributivity property:

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This distributivity property implies κ-distributivity:

$$\bigwedge_{i<\gamma} \bigvee_{j<\gamma} a_{ij} \to \bigvee_{f \in \gamma^\gamma} \bigwedge_{i<\gamma} a_{if(i)}$$

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Example: given an uncountable set A, the theory of injections $f : A \to \mathbf{N}$ does not have models:

$$P_{a,n} \wedge P_{a,m} \vdash \bot \text{ for all } n, m, a \text{ with } n \neq m$$

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As a geometric theory, it is consistent. However, considered as a κ-geometric propositional theory, it is inconsistent.

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We say that the morphisms $h_{\beta,\alpha}$ compose transfinitely, and take the limit projection $f_{\beta,0}$ to be the transfinite composite of $h_{\alpha+1,\alpha}$ for $\alpha < \beta$. There is an exactness condition on $\mathcal{C}$ that we call *transfinite transitivity*: if we have a κ-tree of morphisms of $\mathcal{C}$ where the immediate successors of every node form a jointly covering family, then the transfinite composites of the morphisms along all possible cofinal branches of the tree forms itself a jointly covering family.

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Example/exercise: Any Grothendieck topology is an ω-topology (property T is trivially satisfied).

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- When κ is weakly compact, this is precisely κ-Deligne's theorem. It is essentially equivalent to Karp's completeness theorem for $\mathcal{L}_{\kappa,\kappa}$

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