Fibrations of toposes from extensions of theories

Toposes in Como

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Idea

- For many special constructions of topological spaces, a structure preserving morphism between the presenting structures gives a map between the corresponding spaces. e.g.: a homomorphism $f : K \to L$ between two distributive lattices gives a map *in the opposite direction* between their spectra. The covariance or contravariance of this correspondence is a fundamental property of the construction.

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- In topos theory we can relativize this process: a presenting structure in an elementary topos $\mathcal{E}$ will give rise to a bounded geometric morphism $p \colon \mathcal{F} \to \mathcal{E}$, where $\mathcal{F}$ is the topos of sheaves over $\mathcal{E}$ for the space presented by the structure. Then we commonly find that the covariant or contravariant correspondence mentioned above makes every such p an opfibration or fibration in the 2-category of toposes and geometric morphisms.

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- Using the classifying toposes of geometric theories, we formalize this idea by the notion of fibration of toposes.

Johnstone fibrations in 2-categories

Comprehension 2-category

Suppose $\mathbb{K}$ is a 2-category and $\mathcal{D}$ is a class of bicarrable 1-cells in $\mathbb{K}$ which we shall call "display 1-cells". We form a 2-category $\mathbb{K}_{\mathcal{D}}$ whose

- 0-cells are of the form

$$\begin{array}{c} \overline{X} \\ {\scriptstyle x}\big\downarrow \\ \underline{X} \end{array}$$

where x is a member of class $\mathcal{D}$.

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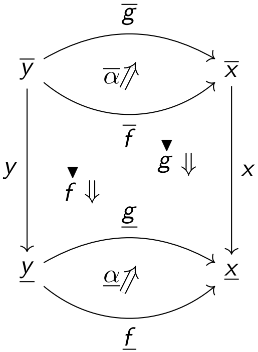
- 1-cells from y to x are of the form $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f} \rangle$

$$\begin{array}{ccc} \overline{Y} & \xrightarrow{\overline{f}} & \overline{X} \\ {\scriptstyle y}\big\downarrow & \overset{\blacktriangledown}{f} \Downarrow & \big\downarrow{\scriptstyle x} \\ \underline{Y} & \xrightarrow[\underline{f}]{} & \underline{X} \end{array}$$

  where $\overset{\blacktriangledown}{f} : x \circ \overline{f} \Rightarrow \underline{f} \circ y$ is an iso 2-cell in $\mathbb{K}$.

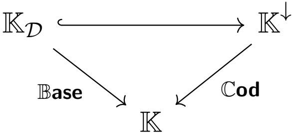
- 2-cells between 1-cells f and g are of the form $\alpha = \langle \overline{\alpha}, \underline{\alpha} \rangle$ where $\overline{\alpha} : \overline{f} \Rightarrow \overline{g}$ and $\underline{\alpha} : \underline{f} \Rightarrow \underline{g}$ are 2-cells in $\mathbb{K}$



in such a way that the obvious diagram of 2-cells commutes.

- Composition: by pasting

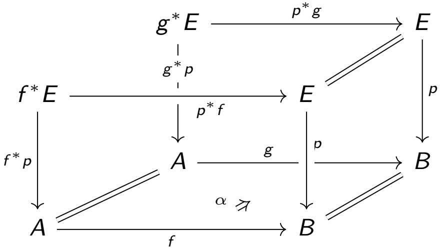
$\mathbb{K}_{\mathcal{D}}$ is a sub 2-category of $\mathbb{K}^{\downarrow}$ and the following diagram of 2-functors commutes.



Johnstone's fibrations in 2-categories

Definition (P. Johnstone, 93)

Suppose $\mathbb{K}$ is a 2-category. A 1-cell $p\colon E \to B$ is an (internal) **fibration** in $\mathbb{K}$ if it is bicarrable and for any 2-cell $\alpha\colon f \Rightarrow g\colon A \rightrightarrows B$ in $\mathbb{K}$, there exists a 1-cell $r(\alpha)\colon \underline{g}^*E \to \underline{f}^*E$, a 2-cell $\widetilde{\alpha}\colon p^*f \circ r(\alpha) \Rightarrow p^*g$, and a 2-cell $\tau(\alpha)\colon f^*p \circ r(\alpha) \Rightarrow g^*p$ satisfying *five axioms*.



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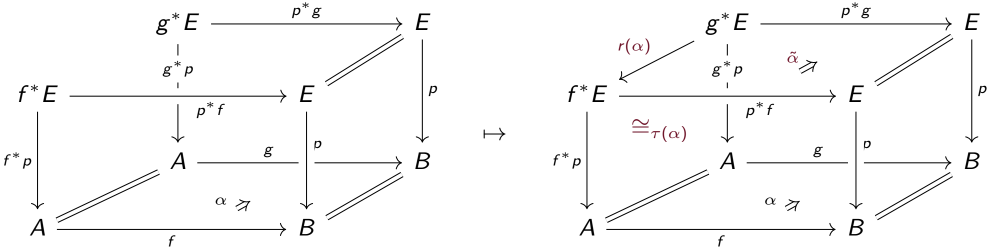

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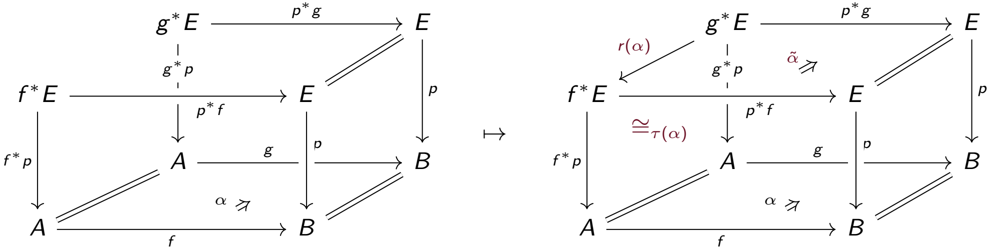
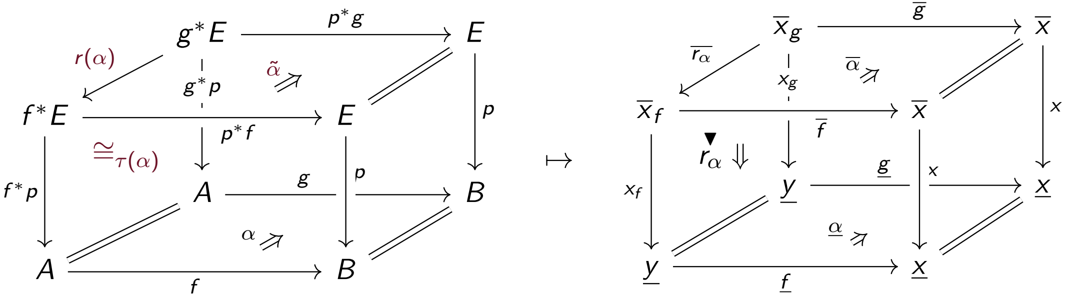


Peter Johnstone. "Fibrations and partial products in a 2-category". In: *Applied Categorical Structures* vol.1.2 (June 1993), pp. 141–179. DOI: 10.1007/BF00880041

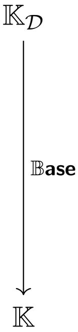
Remark

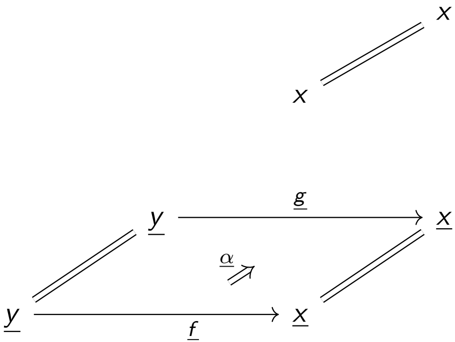
- This definition generalizes the definition of Grothendieck fibration of categories.
- The definition above is equivalent to the representable definition of fibration internal to a 2-category.
- Dually, *opfibrations* are defined by requiring a 1-cell $l(\alpha)\colon f^*E \to g^*E$ in the opposite direction of $r(\alpha)$.
- Johnstone's definition does not require strictness of the 2-category nor the existence of the structure of strict pullbacks and comma objects. Indeed, this definition is most suitable for weak 2-categories such as 2-category of toposes where we do not expect diagrams of 1-cells to commute strictly. This definition is also very flexible in terms of existence of bipullbacks.

Changing the notation ...



Simplifying Johnstone's definition





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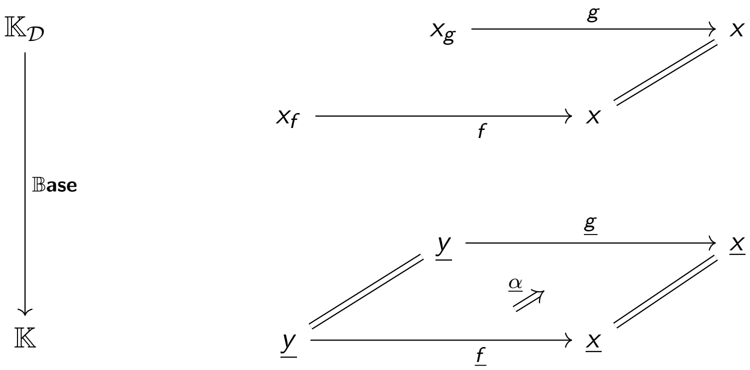

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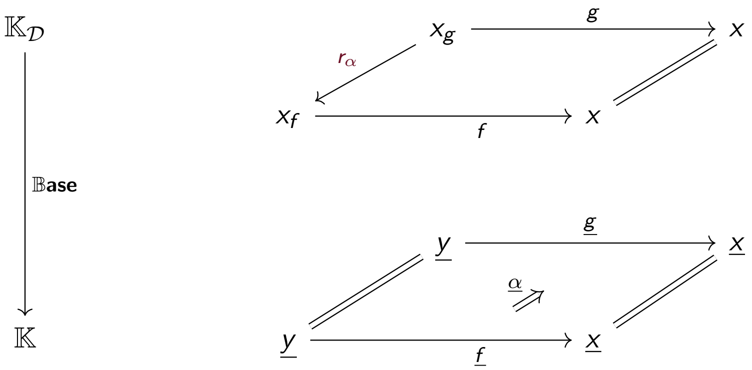

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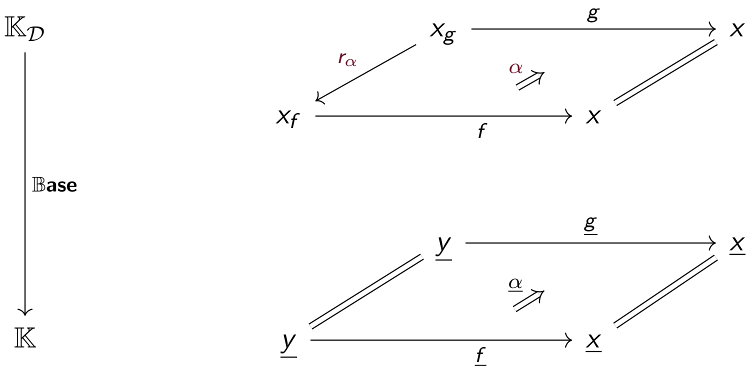
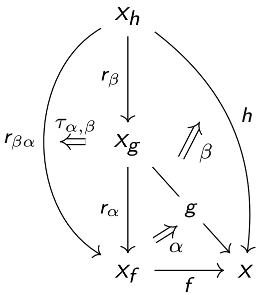


Axioms of Johnstone fibration

1. α lies over $\underline{\alpha}$.

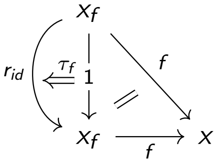
Axioms of Johnstone fibration

2. For any two composable 2-cells $\underline{\alpha} : \underline{f} \Rightarrow \underline{g}$ and $\underline{\beta} : \underline{g} \Rightarrow \underline{h}$ in $\mathscr{K}$ where $\underline{f}, \underline{g}, \underline{h} \colon \underline{y} \to \underline{x}$, the lift of their composition is canonically isomorphic to composition of their lifts α and β in $\mathscr{K}_{\mathcal{D}}$, that is there exists a vertical iso 2-cell $\tau_{\alpha,\beta} \colon r_\alpha \circ r_\beta \Rightarrow r_{\beta\alpha}$ in $\mathscr{K}_{\mathcal{D}}$ such that $\beta \circ (\alpha \centerdot r_\beta) \circ (f \centerdot \tau^{-1}_{\alpha,\beta})$ is the lift of $\underline{\beta} \circ \underline{\alpha}$.



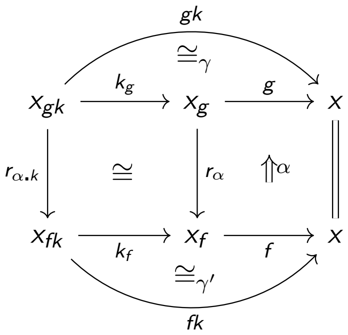
Axioms of Johnstone fibration

3. For any 1-cell $\underline{f}\colon \underline{y} \to \underline{x}$ the lift of identity 2-cell on $\underline{f}$ is canonically isomorphic to the identity 2-cell on the lift f, that is there exists a vertical iso 2-cell $\tau_f \colon 1_f \Rightarrow r_{id_{\underline{f}}}$ in $\mathscr{K}_{\mathcal{D}}$ such that $f \centerdot \tau_f^{-1}$ is the lift of identity 2-cell $id_{\underline{f}}$.



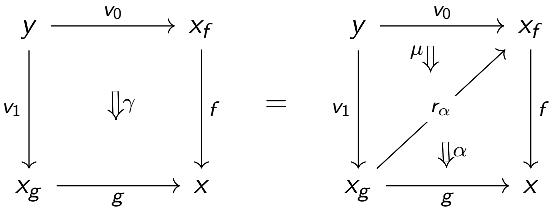
Axioms of Johnstone fibration

4. lift of whiskering of any 2-cell $\underline{\alpha} \colon \underline{f} \to \underline{g} \colon \underline{y} \to \underline{x}$ with any 1-cell $k \colon \underline{z} \to \underline{y}$ is isomorphic, via vertical iso 2-cells γ and γ', to whiskering of the lifts.



Axioms of Johnstone fibration

5. given any pair of vertical 1-cells $v_0 \colon y \to x_f$ and $v_1 \colon y \to x_g$, any 2-cell $\gamma \colon f \circ v_0 \Rightarrow g \circ v_1$ factors through α uniquely, that is there exists a unique 2-cell $\mu \colon v_0 \Rightarrow r_\alpha v_1$ such that the following pasting diagrams are equal.



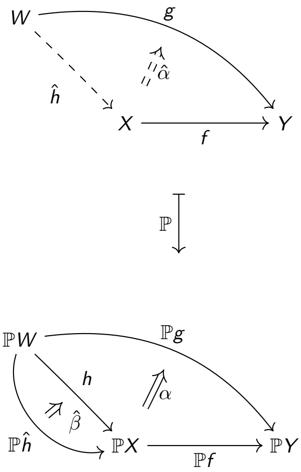
(Weak) cartesian 1-cells

Definition

Suppose $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a 2-functor. A 1-cell $f: X \to Y$ in $\mathbb{X}$ is **cartesian** with respect to $\mathbb{P}$ whenever for each 0-cell W in $\mathbb{X}$ the following commuting square is a *bipullback* diagram in 2-category $\mathfrak{Cat}$ of categories.

$$\begin{array}{ccc} \mathbb{X}(W,X) & \xrightarrow{f_*} & \mathbb{X}(W,Y) \\ {\scriptstyle \mathbb{P}_{W,X}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathbb{P}_{W,Y}} \\ \mathbb{C}(\mathbb{P}W,\mathbb{P}X) & \xrightarrow[\mathbb{P}(f)_*]{} & \mathbb{C}(\mathbb{P}W,\mathbb{P}Y) \end{array}$$

Cartesian 1-cells in elementary terms



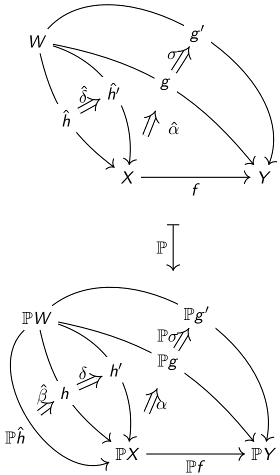
Input data:

1. $g\colon W \to Y$
2. $h\colon \mathbb{P}W \to \mathbb{P}X$
3. iso 2-cell $\alpha\colon \mathbb{P}(f)\circ h \Rightarrow \mathbb{P}(g)$

Output data:

(not necc. unique)

1. $\hat{h}\colon W \to X$
2. iso 2-cell $\hat{\alpha}\colon f\hat{h} \Rightarrow g$
3. iso 2-cell $\hat{\beta}\colon \mathbb{P}(\hat{h}) \Rightarrow h$
4. an equality of 2-cells
$$\alpha \circ (\mathbb{P}(f) \centerdot \hat{\beta}) = \mathbb{P}(\hat{\alpha})$$



Input data:

(1) $\sigma : g \Rightarrow g' : W \rightrightarrows Y$

(2) $\delta \colon h \Rightarrow h' : \mathbb{P}W \rightrightarrows \mathbb{P}X$

(3) iso 2-cells

$\alpha \colon \mathbb{P}(f) \circ h \Rightarrow \mathbb{P}(g)$

$\alpha' : \mathbb{P}(f) \circ h' \Rightarrow \mathbb{P}(g)$

(4) an equality of 2-cells

$\alpha' \circ (\mathbb{P}f \centerdot \delta) = \mathbb{P}(\sigma) \circ \alpha$

Output data:

(1) unique $\hat{\delta} \colon \hat{h} \Rightarrow \hat{h}'$

(2) an equality $\hat{\alpha}' \circ (f \centerdot \hat{\delta}) = \sigma \circ \hat{\alpha}$

(3) an equality $\delta \centerdot (\hat{\beta}) = \hat{\beta}' \circ \mathbb{P}\hat{\delta}$

Cartesian 2-cells

Definition

A 2-cell $\alpha\colon f \Rightarrow g\colon x \to y$ in $\mathbb{X}$ is **cartesian** if it is cartesian as a 1-cell for the functor $\mathbb{P}_{xy}\colon \mathbb{X}(x, y) \to \mathbb{C}(\mathbb{P}x, \mathbb{P}y)$.

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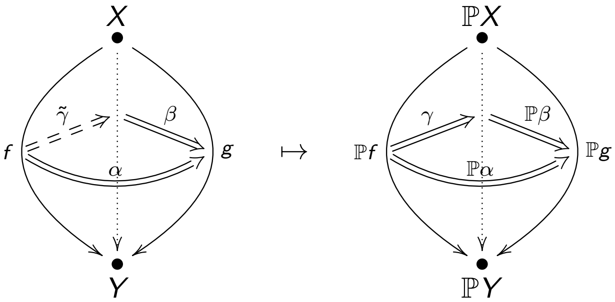
In elementary terms it means a 2-cell $\alpha \colon f \Rightarrow g \colon X \rightrightarrows Y$ is cartesian if for any given 1-cell $e \colon X \to Y$ and 2-cell $\beta \colon e \Rightarrow g$ with $\mathbb{P}\alpha = \mathbb{P}\beta \circ \gamma$ for some 2-cell γ, then there is a unique 2-cell $\tilde{\gamma}$ over γ such that $\alpha = \beta \circ \tilde{\gamma}$.



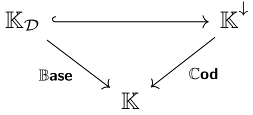
Proposition

A 1-cell $x\colon \overline{x} \to \underline{x}$ in $\mathbb{K}$ is a Johnstone fibration iff

1. every $\underline{f}\colon \underline{y} \to \underline{x} = \mathbb{C}\mathbf{od}(x)$ has a cartesian lift,
2. for every 0-cell y in $\mathbb{K}_{\mathcal{D}}$, the functor

$$\mathbb{C}\mathbf{od}_{y,x}\colon \mathbb{K}_{\mathcal{D}}(y,x) \to \mathbb{K}(\mathbb{C}\mathbf{od}(y), \mathbb{C}\mathbf{od}(x))$$

   is a Grothendieck fibration of categories, and
3. whiskering on the left preserves cartesian 2-cells in $\mathbb{K}_{\mathcal{D}}$ between 1-cells with codomain x.



Relating internal fibrations in 2-categories to fibration of bicategories

Definition

Let $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ be a 2-functor. $\mathbb{X}$ is **fibred over** $\mathbb{C}$ whenever

1. for any $X \in \mathbb{X}$ and $f\colon B \to \mathbb{P}X$ in $\mathbb{C}$, there is a weakly cartesian 1-cell $\widetilde{f}\colon \widetilde{B} \to X$ with $\mathbb{P}\widetilde{f} = f$;
2. $\mathbb{P}$ is locally fibred, i.e. $\mathbb{P}_{XY}\colon \mathbb{X}(X, Y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a Grothendieck fibration of categories for all X, Y in $\mathbb{X}$
3. The horizontal composite of any two cartesian 2-cells is again cartesian.

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This definition is due to Buckley 2014 and he developes a theory of fibred bicategories in Mitchell Buckley. "Fibred 2-categories and bicategories". In: vol. 218 (2014), pp. 1034–1074

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Remark

$\mathbb{K}_{\mathcal{D}}$ is fibred over $\mathbb{K}$ if every 1-cell in $\mathbb{K}_{\mathcal{D}}$ is a fibration in the sense of Johnstone.

2-categories (really bicategories) of toposes

- The 2-category $\mathcal{E}\mathfrak{Top}$ is the 2-category of elementary toposes, geometric morphisms, and natural transformations.
- The 2-category $\mathcal{G}\mathfrak{Top}$ is constructed from 2-category $\mathcal{E}\mathfrak{Top}$ by choosing the class of display morphisms to be bounded geometric morphisms of elementary toposes. So, $\mathcal{G}\mathfrak{Top} = \mathcal{E}\mathfrak{Top}_{\mathcal{D}}$ where $\mathcal{D}$ is the class of bounded geometric morphisms of elementary toposes.

$$\begin{array}{ccc} \mathcal{G}\mathfrak{Top} & \hookrightarrow & \mathcal{E}\mathfrak{Top}^{\downarrow} \\ & {\scriptstyle \mathbb{B}\mathbf{ase}} \searrow \quad \swarrow {\scriptstyle \mathbb{C}\mathbf{od}} & \\ & \mathcal{E}\mathfrak{Top} & \end{array}$$

- A bounded geometric morphism $p \colon \mathscr{E} \to \mathscr{S}$ is a fibration of toposes if it is a fibration 0-cell in $\mathcal{G}\mathfrak{Top}$.

Classifying toposes as representing objects

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- To a geometric morphism $\langle f^*, f_* \rangle \colon \mathscr{F} \to \mathscr{E}$ of $\mathscr{S}$-toposes it assigns the functor $f^* \colon \mathbb{T}\text{-}\mathbf{Mod}\text{-}\mathscr{E} \to \mathbb{T}\text{-}\mathbf{Mod}\text{-}\mathscr{F}$.

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- Note that $\mathbb{T}\text{-}\mathbf{Mod}\text{-}(f \circ g) \cong (\mathbb{T}\text{-}\mathbf{Mod}\text{-}f) \circ (\mathbb{T}\text{-}\mathbf{Mod}\text{-}g)$
- The classifying topos $\mathscr{S}[\mathbb{T}]$ of a geometric theory/context $\mathbb{T}$ can be seen as a representing object for this pseudofunctor, i.e.

$$\mathcal{B}\mathfrak{Top}/\mathscr{S}(\mathscr{E}, \mathscr{S}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathbf{Mod}\text{-}\mathscr{E}$$

naturally in $\mathscr{E}$.

Fibrations of toposes from extension of theories

- Fix an elementary topos $\mathcal{S}$. Every geometric theory/ context $\mathbb{T}$ gives rise to an indexed category over $\underline{\mathbb{T}} : \mathcal{B}\mathfrak{Top}/\mathcal{S}$, where

$$\underline{\mathbb{T}}(\mathscr{E}) :=\mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathscr{E}) = \text{category of models of } \mathbb{T} \text{ in } \mathscr{E}$$

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- Of course not all elephant theories arise from contexts. For instance, given a bounded geometric morphism $p \colon \mathscr{E} \to \mathscr{S}$ and a context extension $U \colon \mathbb{T}_1 \to \mathbb{T}_0$ is a context extension and M is a strict model of context $\mathbb{T}$ in base topos $\mathcal{S}$, then $\mathbb{T}_1/M$ is an elephant theory but not a context, where

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- Certain elephant theories are geometric and have classifying toposes. $\underline{\mathbb{T}}$ and $\underline{\mathbb{T}_1/M}$ are such examples.

Theorem (Vickers, 2017)

Suppose $U : \mathbb{T}_1 \to \mathbb{T}_0$ is a context extension. For any model M of $\mathbb{T}_0$ in a (base) topos $\mathcal{S}$, $\mathcal{S}[\mathbb{T}_1/M]$ is an $\mathcal{S}$-topos, and moreover, for any geometric (not necessarily bounded) morphism $\underline{f} : \mathcal{A} \to \mathcal{S}$, the classifying topos $\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M]$ is got by bi-pullback of $\mathcal{S}[\mathbb{T}_1/M]$ along $\underline{f}$:

$$
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\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M] & \xrightarrow{\;\overline{f}\;} & \mathcal{S}[\mathbb{T}_1/M] \\
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Chevalley fibrations

- Suppose $\mathbb{K}$ is a 2-category with finite (strict) PIE-limits, in other words those reducible to Products, Inserters and Equifiers.
- This is enough to guarantee existence of all strict comma objects.

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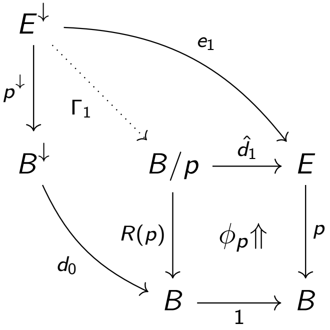
- Suppose B is an object of $\mathbb{K}$, and p is a 0-cell in the strict slice 2-category $\mathbb{K}/B$. p is a **Chevalley fibration** if the 1-cell Γ_1 has a right adjoint Λ_1 with counit an identity in the 2-category $\mathbb{K}/B$.



Chevalley fibrations

- Dually one defines Chevalley **opfibrations** as 1-cells $p\colon E \to B$ for which the morphism $\Gamma_0\colon E^{\downarrow} \to p/B$ has a left adjoint Λ_0 with identity unit.

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- Street 1974 characterizes Chevalley fibrations as pseudo-algebras of a slicing KZ 2-monads on 2-categories in Ross Street. "Fibrations and Yoneda's lemma in a 2-category". In: *Lecture Notes in Math., Springer, Berlin* vol.420 (1974), pp. 104–133.

Fibrational extensions of contexts

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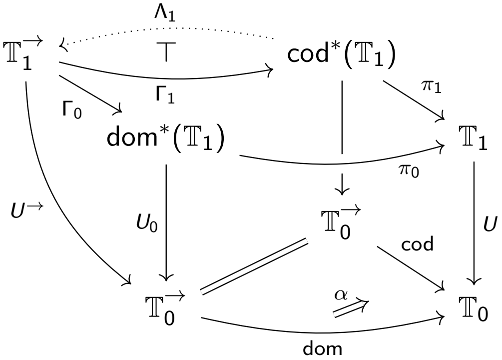
- Using this fact, and since comma objects exists in $\mathfrak{Con}$, we reformulate the notion of Chevalley fibration in $\mathfrak{Con}$.

Fibrational extensions of contexts

- An extension map is called **fibrational** if Γ_1 has a right adjoint with identity counit.



Main theorem

Theorem

If $U : \mathbb{T}_1 \to \mathbb{T}_0$ is a (op)fibrational extension of contexts, and M is any model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$, then $p : \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ is an (op)fibration of toposes.

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Local homeomorphism of toposes as opfibration

- For $\mathcal{S}$ a bounded $\mathcal{S}_0$ topos, and $\mathbb{T}_0 = \mathbb{O}$ and $\mathbb{T}_1$ the extended context of $\mathbb{T}_0$ with a fresh edge from terminal to the unique node of $\mathbb{T}_0$.

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- And a bipullback of toposes

$$\begin{array}{ccc}
\mathcal{S}/M \simeq \mathcal{S}[\mathbb{T}_1/M] & \longrightarrow & \mathcal{S}_0[X, x] = \mathcal{S}_0[X][\mathbb{T}_1/X] \\
{\scriptstyle M^* p}\downarrow & & \downarrow{\scriptstyle p} \\
\mathcal{S} & \xrightarrow[M]{\qquad\qquad} & \mathcal{S}_0[X]
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- M^*p is a fibration of toposes.

Spectrum of Boolean algebras

- For $\mathcal{S}$ a bounded $\mathcal{S}_0$ topos, and $\mathbb{T}_0$ = context of Boolean algebras and $\mathbb{T}_1$ the extended context of Boolean algebra with a prime filter
- We get a context extension map $\mathbb{T}_1 \to \mathbb{T}_0$ which is a fibration.
- And a bipullback of toposes

$$\begin{array}{ccc} Spec(B) & \longrightarrow & \mathcal{S}[\mathbb{T}_1/B] \\ {\scriptstyle M^*p}\downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[B]{} & \mathcal{S} \end{array}$$

- The points of $\mathcal{S}[\mathbb{T}_1/B]$ are pairs (B, F) where F is an internal prime filter of B in topos $\mathcal{S}$. "every fibrewise Stone bundle is a fibration."

Other examples

- Internal Algebraic dcpos as opfibrations
- Spectral spaces as fibrations
- SFP domains as bifibrations
- Internal groups equipped with an action as fibrations
- Intenral categories equipped with a torsor as opfibrations
- Internal modules as bifibrations
- Bag domains as opfibrations
- ...

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End!

Thank you for your attention!