

Models of Univalence in Toposes

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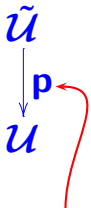
Plan

- ▶ Universes in toposes
- ▶ Univalence: extensionality for universes
- ▶ Identity types from intervals in toposes
- ▶ Univalent universes from tiny intervals
- ▶ Conclusion

Universes in a topos \mathcal{E}

Types whose elements denote types are crucial for the expressive power of Martin-Löf Type Theory (MLTT).

Modelling MLTT in toposes leads one to consider universe objects



a family of \mathcal{E} -objects, indexed by the generalized elements of u

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$$\begin{array}{ccc} & & \tilde{\mathcal{U}} \\ & & \downarrow \mathbf{p} \\ \Gamma & \xrightarrow{u} & \mathcal{U} \end{array}$$

a generalized element u at stage $\Gamma \in \mathcal{E}$

Universes in a topos \mathcal{E}

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A commutative square diagram with vertices \cdot , $\tilde{\mathcal{U}}$, Γ , and \mathcal{U} . The top edge is a horizontal arrow from \cdot to $\tilde{\mathcal{U}}$. The right edge is a vertical arrow from $\tilde{\mathcal{U}}$ to \mathcal{U} labeled \mathbf{p} . The bottom edge is a horizontal arrow from Γ to \mathcal{U} labeled u . The left edge is a vertical arrow from \cdot to Γ labeled $\mathbf{el} u$. A right-angle symbol is at the top-right corner of the square. A red arrow points from the text box below to the $\mathbf{el} u$ label.

a generalized element u at stage $\Gamma \in \mathcal{E}$
gives an object $\mathbf{el} u$ in \mathcal{E}/Γ by pulling back \mathbf{p} along u :
 u is a **code** for the object $\mathbf{el} u$

Universes in a topos \mathcal{E}

Bénabou; Hofmann-Streicher: can transfer a notion of size in **Set** (Grothendieck universe) to give a corresponding universe in any topos bounded over **Set** (Grothendieck topos).

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Assumption: will only consider toposes \mathcal{E} equipped with a sequence of universes $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \dots$ that are

- ▶ nested: \mathcal{S}_n has a code in \mathcal{S}_{n+1}
- ▶ internal full subtoposes, closed under taking dependent products, etc. [details omitted]

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However, there's more to universes than just questions of size. . .

Extensionality for universes

An **extensionality principle** for some type of data (functions, pairs, trees, ...) gives us a handle on reasoning about equalities between items of data.

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An extensionality principle for some type of data (functions, pairs, trees, ...) gives us a handle on reasoning about equalities between items of data.

What about universes? In that case the data are codes for types.

$$\begin{array}{ccc} \cdot & \longrightarrow & \tilde{\mathcal{U}} \\ \text{el } u \downarrow & \lrcorner & \downarrow \mathbf{p} \\ \Gamma & \xrightarrow{u} & \mathcal{U} \end{array}$$

How do we explain when two such codes are equal?

$$\forall u, v \in_{\Gamma} \mathcal{U}, \quad u = v \Leftrightarrow ?$$

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$$\forall u, v \in_{\Gamma} \mathcal{U}, \quad u = v \Leftrightarrow \text{el } u = \text{el } v$$

Can hold for universes of sets ($\mathcal{E} = \mathbf{Set}$), but is mathematically uninteresting – in general, we should only care about objects up to **isomorphism** (\cong), not equality ($=$).

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$$\forall u, v \in_{\Gamma} \mathcal{U}, \quad u = v \Leftrightarrow \text{el } u \cong \text{el } v$$

A too strong, “skeletal” property – hard to achieve and anyway it doesn’t take account of the different ways that u and v can be isomorphic.

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$$\forall u, v \in_{\Gamma} \mathcal{U}, \quad \text{ua} : (u \sim v) \simeq (\text{el } u \simeq \text{el } v)$$

Key idea (Voevodsky's univalence axiom): replace predicate “ $u = v$ ” by **identity types** $u \sim v$ (generalized elements of which are like proofs of equality of u and v) and compare it with an object of **equivalences** (\simeq) between $\text{el } u$ and $\text{el } v$ (\simeq is itself defined in terms of \sim , rather than $=$).

While discussing **ua**, allow me to blur the distinction between

syntax	&	semantics
Γ context		$\Gamma \in \mathcal{E}$
$\Gamma \vdash A$ type		$\begin{array}{c} \bullet \\ \downarrow_A \\ \Gamma \end{array} \in \mathcal{E}/\Gamma$
$\Gamma \vdash a : A$		$\begin{array}{ccc} \Gamma & \xrightarrow{a} & \bullet \\ & \searrow_{\text{id}} & \downarrow_A \\ & & \Gamma \end{array}$
$\prod_{x:A} B(x)$		local exponentials
etc. . .		

and to ride rough shod over questions of how to deal with “strictness” (semantics of substitution).

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Identity types

$$\frac{a : A \quad a' : A}{a \sim_A a' \text{ type}}$$

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(Adapting a trick due to **Lumsdaine**, given the above, we can always redefine the **transport operation** $*$ to ensure there is a proof of $\text{refl} * a \sim_A a$ and hence, by **Coquand-Danielsson**, of **identity induction**.)

Identity types

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$$\frac{p : a \sim_A a'}{\text{contr} : (a, \text{refl}) \sim_{\Sigma_{x:A}(a \sim_A x)} (a', p)}$$
$$\frac{x : A \vdash B(x) \text{ type} \quad p : a \sim_A a' \quad b : B(a)}{p * a : B(a')}$$

\mathcal{E} 's extensional equality ($=$) gives identity types

$$a \sim_A a' \triangleq \{x : 1 \mid a = a' : A\}$$

that are too “thin” to satisfy **ua** (unless \mathcal{E} is degenerate). A more intensional notion of equality is needed for **ua**.

Homotopical view of identity types

space	A type
point	$a : A$
path	$p : a \sim_A a'$
homotopy	$\pi : p \sim_{(a \sim_A a')} p'$
	\vdots

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Voevodsky et al [[arXiv:1211.2851v4](https://arxiv.org/abs/1211.2851v4)]: the topos of simplicial sets (in classical ZFC) provides a concrete instance of this analogy and has univalent universes w.r.t. usual Kan simplicial path types.

Homotopical view of identity types

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Cohen-Coquand-Huber-Mörtberg (CCHM) [[arXiv:1611.02108](https://arxiv.org/abs/1611.02108)]: a presheaf topos of “cubical sets” (in IZF, say) that provides a constructive model of univalence.

There are now several variations on cubical sets.

Orton-AMP [[arXiv:1712.04864](https://arxiv.org/abs/1712.04864)], **Licata-Orton-AMP-Spitters** [[arXiv:1801.07664](https://arxiv.org/abs/1801.07664)]: an elementary axiomatic treatment of what makes these models of univalence tick.

Given $0, 1 : \mathbf{1} \rightrightarrows \mathbb{I}$ in some topos \mathcal{E} ,
what is required of the “interval” object \mathbb{I} to make
 $a \sim_A a' \triangleq \{p : A^{\mathbb{I}} \mid p0 = a \wedge p1 = a'\}$
an identity type for which there is a univalent universe?

- ▶ Can satisfy $\frac{a : A}{\mathbf{refl} : a \sim_A a}$ by taking $\mathbf{refl} = \lambda i. a$

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- ▶ Can satisfy $\frac{p : a \sim_A a'}{\text{contr} : (a, \text{refl}) \sim_{\Sigma_{x:A}(a \sim_A x)} (a', p)}$ by taking
 $\text{contr} = \lambda i. (p(i), \lambda j. p(i \sqcap j))$
 provided \mathbb{I} has a **connection** $_ \sqcap _ : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I}$, a weak form
 of binary minimum satisfying some simple axioms.

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► To get transport operations

$_ * _ : (p : a \sim_A a') \rightarrow B(a) \rightarrow B(a')$ for families
 $x : A \vdash B(x)$, we equip them with extra structure –
CCHM fibration structure...

CCHM fibration structure

... is a form of (uniform) **Kan-filling** operation with respect to “cofibrations”:

Assumption: \mathcal{E} comes equipped with a class of monos, called **cofibrations**, generated under pullback from a generic one $\mathbf{1} \twoheadrightarrow \mathbf{Cof}$ (where $\mathbf{Cof} \twoheadrightarrow \mathbf{\Omega}$) satisfying. . . [axioms t.b.a.]

CCHM fibration structure

Given a family of types $A : \Gamma \rightarrow \mathcal{S}$,
a CCHM fibration structure $\alpha : \mathbf{Fib} A$ maps

path in Γ	$p : \Gamma^{\mathbb{I}}$
cofibrant partial path over p	$f : \prod_{i:\mathbb{I}} (\varphi \rightarrow A(p i))$ with $\varphi : \mathbf{Cof}$
extension of f at 0	$a_0 : A(p 0)$ with $f 0 \uparrow a_0$

to

extension of f at 1	$a_1 : A(p 1)$ with $f 1 \uparrow a_1$
-------------------------	--

where **extension relation** for $\varphi : \mathbf{Cof}$, $f : \varphi \rightarrow \Gamma$ and $x : \Gamma$ is

$f \uparrow x \triangleq \prod_{u:\varphi} (f u = x)$ “ f agrees with x where defined”

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(In particular, so long as $\mathbf{false} : \mathbf{Cof}$, then each $\alpha : \mathbf{Fib} A$ induces a transport function $_ * _ : \prod_{p:\Gamma^{\mathbb{I}}} (A\ 0 \rightarrow A\ 1)$, namely $p * a = \pi_1(\alpha\ p\ \mathbf{false}!\ (a, !))$.)

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Some simple closure properties of \mathbf{Cof} enable one to prove that the existence of fibration structure is preserved under forming Σ -types, Π -types, path types,...

What about universes of fibrations?

Tiny interval

$$\mathcal{F}(\Gamma) \triangleq \sum_{A:\Gamma \rightarrow \mathcal{S}} \mathbf{Fib} A$$

set of **CCHM fibrations** over an object $\Gamma \in \mathcal{E}$

Theorem. If interval $\mathbb{I} \in \mathcal{E}$ is **tiny**, i.e. $(_)^\mathbb{I}$ has a right adjoint (denoted $\checkmark : \mathcal{E} \rightarrow \mathcal{E}$), then $\mathcal{F}(_) : \mathcal{E}^{\text{op}} \rightarrow \mathbf{Set}$ is representable and we get a universe of CCHM fibrations.

Theorem generalizes unpublished work of **Coquand & Sattler** for the case \mathcal{E} is a presheaf topos.

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When $\mathcal{E} = \mathbf{Set}^{\square^{\text{op}}}$, the topos of cubical sets,

the category \square has finite products

and the interval in \mathcal{E} is representable: $\mathbb{I} = \square(_, I)$.

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Hence the path functor $(_)^\mathbb{I} : \mathbf{Set}^{\square^{\text{op}}} \rightarrow \mathbf{Set}^{\square^{\text{op}}}$ is $(_ \times I)^*$

and so $(_)^\mathbb{I}$ not only has a left adjoint $(_ \times \mathbb{I})$, but also a right adjoint.

Classifying fibration

The Theorem says that one can use tinytness of \mathbb{I} to construct a

classifying fibration $(\mathcal{U} \xrightarrow{\pi_1} \mathcal{S}, \nu : \mathbf{Fib} \pi_1) \in \mathcal{F}(\mathcal{U})$ satisfying:

for all $(\Gamma \xrightarrow{A} \mathcal{S}, \alpha : \mathbf{Fib} A) \in \mathcal{F}(\Gamma)$

there is a unique $\ulcorner A, \alpha \urcorner : \Gamma \rightarrow \mathcal{U}$ such that

$$(A, \alpha) = \mathcal{F}(\ulcorner A, \alpha \urcorner)(\pi_1, \nu)$$

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Given $(\Gamma \xrightarrow{A} \mathcal{S}, \alpha : \mathbf{Fib} A) \in \mathcal{F}(\Gamma)$

A commutative diagram illustrating the relationship between various categories and functors. The diagram consists of the following nodes and arrows:

- Top-left node: $\Gamma^{\mathbb{I}}$
- Top-right node: $\tilde{\mathcal{S}}$
- Bottom-left node: $\mathcal{S}^{\mathbb{I}}$
- Bottom-right node: \mathcal{S}

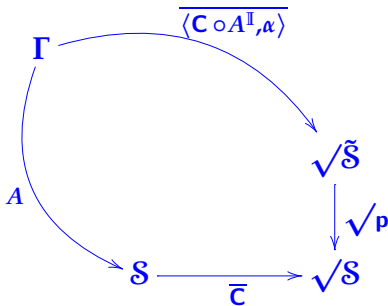
The arrows are:

- A curved arrow from $\Gamma^{\mathbb{I}}$ to $\tilde{\mathcal{S}}$ labeled $\langle C \circ A^{\mathbb{I}}, \alpha \rangle$.
- A curved arrow from $\Gamma^{\mathbb{I}}$ to $\mathcal{S}^{\mathbb{I}}$ labeled $A^{\mathbb{I}}$.
- A horizontal arrow from $\mathcal{S}^{\mathbb{I}}$ to \mathcal{S} labeled C .
- A vertical arrow from $\tilde{\mathcal{S}}$ to \mathcal{S} labeled P .

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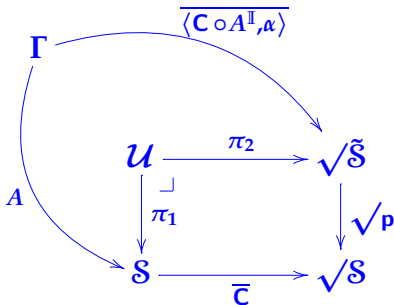
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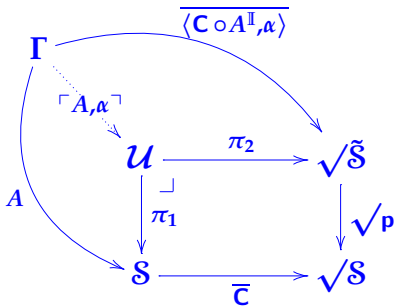


$\mathcal{U} \triangleq$ pullback of \bar{C} and \sqrt{p}

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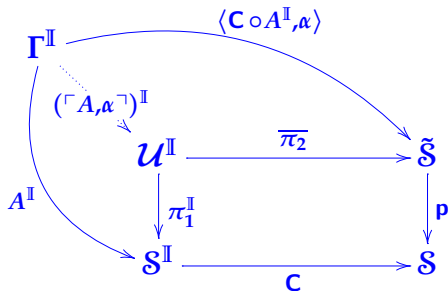
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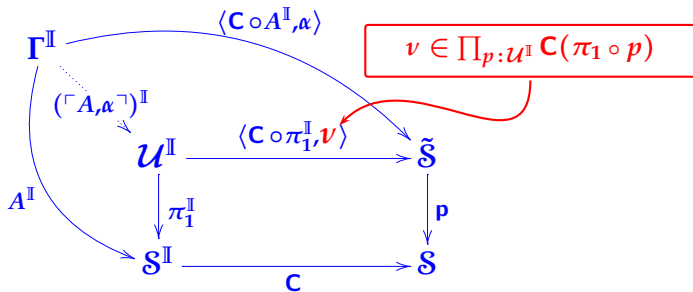
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uniqueness of $\lceil A, \alpha \rceil$ follows from uniqueness property of the pullback square

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and from that a universe, by pulling back:

$$\begin{array}{ccc} \tilde{\mathcal{U}} & \longrightarrow & \tilde{\mathcal{S}} \\ \downarrow & \lrcorner & \downarrow \mathbf{p} \\ \mathcal{U} & \xrightarrow{\pi_1} & \mathcal{S} \end{array}$$

Theorem. Universe $\tilde{\mathcal{U}} \rightarrow \mathcal{U}$ is closed under Π -types, \sim -types and inductive types (e.g. Σ) if \mathbf{Cof} satisfies

$$\mathbf{false} \in \mathbf{Cof}$$

$$\forall i : \mathbb{I}, \varphi : \mathbf{Cof}, (\varphi \vee i = 0 \in \mathbf{Cof}) \wedge (\varphi \vee i = 1 \in \mathbf{Cof})$$

What about univalence of $\tilde{\mathcal{U}} \rightarrow \mathcal{U}$?

Homotopical view of identity types

space	A type
point	$a : A$
path	$p : a \sim_A a'$
homotopy	$\pi : p \sim_{(a \sim_A a')} p'$
	\vdots

One of Voevodsky's key insights: the role of **contractibility**

$$\mathbf{isContr} A \triangleq \sum_{a:A} \prod_{a':A} (a \sim_A a')$$

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$$\mathbf{isEquiv} f \triangleq \prod_{b:B} \mathbf{isContr} (\sum_{a:A} (f a \sim_B b))$$

$$A \simeq B \triangleq \sum_{f:B^A} \mathbf{isEquiv} f$$

$$\mathbf{UA}(\mathcal{U}, \mathbf{el}) \triangleq \prod_{u,v:\mathcal{U}} \mathbf{isEquiv}(u \sim_{\mathcal{U}} v \rightarrow \mathbf{el} u \simeq \mathbf{el} v)$$

Univalence

Theorem. For any topos \mathcal{E} with \mathbb{I} & \mathbf{Cof} satisfying assumptions so far, there is a term of type $\mathbf{UA}(\mathcal{U}, \mathbf{el})$ if

\mathbf{Cof} is closed under $(\forall i : \mathbb{I})_$ and satisfies the **isomorphism extension axiom**...

In this case $\mathcal{U} \rightarrow \mathbf{1}$ is a fibration and $(\mathcal{U}, \mathbf{el})$ is univalent.

Proof is non-trivial.

Isomorphism extension axiom

$$\mathbf{iea} : \prod_{A:\mathcal{S}} \mathbf{Ext}(\sum_{B:\mathcal{S}} (A \cong B))$$

(where $\mathbf{Ext} \Gamma \triangleq \prod_{\varphi:\mathbf{Cof}} \prod_{f:\varphi \rightarrow \Gamma} \sum_{x:\Gamma} (f \uparrow x)$)

$$\begin{array}{ccc} \Delta & \xrightarrow{\varphi \text{ (cofibrant)}} & \Gamma & \xrightarrow{A} & \mathcal{S} \\ & \searrow & & \nearrow & \\ & & B \cong A \circ \varphi & & \\ & \text{---} & & \text{---} & \\ & & B & & \end{array}$$

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$$\mathbf{iea} : \prod_{A:\mathcal{S}} \mathbf{Ext}(\sum_{B:\mathcal{S}} (A \cong B))$$

(where $\mathbf{Ext} \Gamma \triangleq \prod_{\varphi:\mathbf{Cof}} \prod_{f:\varphi \rightarrow \Gamma} \sum_{x:\Gamma} (f \uparrow x)$)

$$\begin{array}{ccc} \Delta & \xrightarrow{\varphi \text{ (cofibrant)}} & \Gamma & \xrightarrow[A]{\cong} & \mathcal{S} \\ & \searrow & & & \nearrow \\ & & & & \mathcal{S} \end{array}$$

$B = A' \circ \varphi$
 B

Isomorphism extension axiom

$$\mathbf{iea} : \prod_{A:\mathcal{S}} \mathbf{Ext}(\sum_{B:\mathcal{S}} (A \cong B))$$

(where $\mathbf{Ext} \Gamma \triangleq \prod_{\varphi:\mathbf{Cof}} \prod_{f:\varphi \rightarrow \Gamma} \prod_{x:\Gamma} (f \uparrow x)$)

Theorem. In a presheaf topos $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$, \mathbf{Cof} satisfies **iea** if for each $X \in \mathbf{C}$ and $S \in \mathbf{Cof}(X) \subseteq \Omega(X)$, the sieve S is a decidable subset of \mathbf{C}/X .

(So with classical meta-theory, it always holds for presheaf toposes.)

For the CCHM topos of cubical sets $\mathbf{Set}^{\square^{\text{op}}}$, the authors make a particular choice for \mathbf{Cof} that does have this decidability property (and which is closed under $(_)\mathbb{I}$).

Summary of assumptions

- ▶ Elementary topos \mathcal{E} with universes $\Omega : \mathcal{S}_0 : \mathcal{S}_1 : \mathcal{S}_2 : \dots$
- ▶ “Interval” object \mathbb{I} (in \mathcal{S}_0) which has distinct end-points & connection operation (& for convenience, a reversal operation) and which is tiny.
- ▶ Universe of “cofibrant” propositions $\mathbf{Cof} \multimap \Omega$ containing **false**, $i = 0$, $i = 1$, closed under $_ \vee _$ and $\forall (i : \mathbb{I}) _$ and satisfying the isomorphism extension axiom.

Conclusion

- ▶ Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets, because the path functor is fibered over \mathcal{E} and we can use internal language to describe many of the constructions on the way to a univalent universe. . .

Conclusion

- ▶ Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets. because the path functor is fibered over \mathcal{E} and we can use internal language to describe many of the constructions on the way to a univalent universe. . .

. . . but not all of them: tinytness does not internalize! (so neither does our universe construction)

Licata-Orton-AMP-Spitters use type theory enriched with a modality for global properties (“**crisp type theory**”) in order to express the whole construction with a type-theoretic language.

Conclusion

- ▶ Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- ▶ The axiomatic approach helps one see the wood from the trees in existing models and to find new ones
(cf. recent work of Uemura using the effective topos)

Conclusion

- ▶ Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- ▶ The axiomatic approach helps one see the wood from the trees in existing models and to find new ones
- ▶ Nevertheless, some of the theorems on the way to univalence/fibrancy are delicate and hard work!

We find the use of an interactive theorem proving system (Agda-flat) invaluable for developing and checking the proof – e.g. see [\[doi.org/10.17863/CAM.21675\]](https://doi.org/10.17863/CAM.21675)

Conclusion

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- ▶ The axiomatic approach helps one see the wood from the trees in existing models and to find new ones
- ▶ Nevertheless, some of the theorems on the way to univalence/fibrancy are delicate and hard work!
Are there simpler models of univalence? (must be non-truncated to qualify for our attention)
E.g. can one avoid Kan-filling in favour of a (weak) notion of path composition?
Why only presheaf toposes?

Conclusion

- ▶ Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- ▶ The axiomatic approach helps one see the wood from the trees in existing models and to find new ones
- ▶ Nevertheless, some of the theorems on the way to univalence/fibrancy are delicate and hard work!
- ▶ Further reading:
 - I. Orton and A. M. Pitts, *Axioms for Modelling Cubical Type Theory in a Topos* [[arXiv:1712.04864](https://arxiv.org/abs/1712.04864)]
 - D. R. Licata, I. Orton, A. M. Pitts and B. Spitters, *Internal Universes in Models of Homotopy Type Theory* [[arXiv: 1801.07664](https://arxiv.org/abs/1801.07664)]

Thank you for your attention!