Nonconnective structures

Toposes in Como

Università dell'Insubria

Mauro Porta (Université de Strasbourg)

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Plan of the talk

- Motivations: HKR theorems in analytic geometry
- Background on derived analytic geometry
- Nonconnective structures

Let k be a field and let A be a k-algebra. The ith Hochschild homology group is

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Theorem (HKR)

Let k be a field of characteristic zero and let A be a smooth k-algebra. Then there is a multiplicative isomorphism

$$\mathrm{HH}_*(A) \simeq \Lambda^* \Omega^1_{A/k}.$$

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Consequence: the correct guess becomes

$$A \otimes^L_{A \otimes_k A} A \simeq \mathrm{Sym}_A(\mathbb{L}_{A/k}[1]).$$

We would like the above equivalence to be multiplicative.

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$$\mathrm{HH} \colon \mathrm{sCAlg}_k \longrightarrow \mathrm{sCAlg}_k$$

sending A to $A \otimes^L_{A \otimes_k A} A$ and

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Strategy: prove that these functors have right adjoints and that they are equivalent.

Problem: the right adjoints do not exist!

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\mathrm{cdga}_k & \xrightarrow{\mathrm{HH},\mathrm{DR}} & \mathrm{cdga}_k \\
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Proposition

The functors $\mathrm{HH}, \mathrm{DR} \colon \mathrm{cdga}_k \to \mathrm{cdga}_k$ *admit right adjoints.*

The right adjoint in the de Hochschild case

Since we work in the commutative setting, we have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{cdga}_k}(\mathrm{HH}(A), B) &\simeq \mathrm{Map}_{\mathrm{cdga}_k}(A \otimes^{L}_{A \otimes_k A} A, B) \\
&\simeq \mathrm{Map}(A, B) \times_{\mathrm{Map}(A,B) \times \mathrm{Map}(A,B)} \mathrm{Map}(A, B) \\
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Remark: $B \times_{B \times B} B \simeq B \otimes_k (k \times_{k \times k} k) \simeq B \oplus B[-1]$.

The right adjoint in the de Rham case

Fix $f : A \to B$. We have a fiber sequence

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{cdga}_k/A}(\mathrm{Sym}_A(\mathbb{L}_A[1]), B) & \longrightarrow & \mathrm{Map}_{\mathrm{cdga}_k}(\mathrm{Sym}_A(\mathbb{L}_A[1]), B) \\ \downarrow & & \downarrow \\ \{*\} & \xrightarrow{f} & \mathrm{Map}_{\mathrm{cdga}_k}(A, B) \end{array}$$

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Now

$$\begin{aligned} \mathrm{Map}_{\mathrm{cdga}_k/A}(\mathrm{Sym}_A(\mathbb{L}_A[1]), B) &\simeq \mathrm{Map}_{A\text{-}\mathrm{Mod}}(\mathbb{L}_A[1], B) \\ &\simeq \mathrm{Der}_k(A; f_* B[-1]). \end{aligned}$$

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Observe: there is a multiplicative zig-zag

$$k \oplus k[-1] \longleftarrow \mathrm{Sym}_k(k[-1]) \longrightarrow k \times_{k \times k} k.$$

When $\mathrm{char}(k) = 0$, these are equivalences.

The goal

The HKR theorem can be reformulated as follows:

Theorem

*Let X be a scheme over k. There is an equivalence of **derived** schemes*

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Our goal: prove the analytic version.
This is an ongoing project with J. António and F. Petit.

Background on derived analytic geometry

Let's start with a naïf guess:

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So one might expect to obtain the category of derived analytic spaces as a full subcategory of the ∞-category of simplicially ringed spaces (or, better, ∞-topoi). For example the next definition sounds reasonable:

Definition (Wrong)

A derived analytic space is a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, where:

1. $\mathcal{X}$ is the topos of sheaves on either a $\mathbb{C}$-analytic space or on the quasi-étale site of a Berkovich space;
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There are many (equivalent) ways of explaining the problem with the above definition:

1. the associated deformation theory is not what we expect;
2. Lurie's representability theorem fails;
3. the category of derived analytic spaces (in the above sense) is not closed under pushouts along closed immersions.

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Definition

Let $(X, \mathcal{O}_X)$ be an analytic space and let F be a (discrete) coherent sheaf on X. An analytic derivation on X with values in F is a derivation $\mathcal{O}_X \to F$ which furthermore satisfies

$$d(f(x)) = f'(x)dx$$

for every $x \in \mathcal{O}_X(U)$ and every analytic function $f\colon \mathbf{A}^1 \to \mathbf{A}^1$.

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To do this in a sufficiently canonical way, we need to endow the structure sheaf of our derived analytic spaces with an *extra analytic structure*.

Analytic structure sheaves

Essentially two different approaches to formalize the idea:

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- Interpret $\mathcal{O}_{\mathcal{X}}$ as a sheaf of simplicial algebras equipped with an axiomatic holomorphic functional calculus (Lurie, P., P. - Yu).

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Here it is a recap on functional calculus. Let A is a commutative Banach $\mathbb{C}$-algebra. Any element $a \in A$ determines a solid arrow

$$\begin{array}{ccc} \mathbb{C}[z] & \xrightarrow{a} & A \\ \downarrow & \nearrow & \\ \mathcal{H}(U) & & \end{array}$$

If $U \subset \mathbb{C}$ is an open subset and $\mathcal{H}(U)$ denotes the ring of holomorphic functions on U, we have a restriction $\mathbb{C}[z] \to \mathcal{H}(U)$.

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Answer: Let $\sigma(a) \subset \mathbb{C}$ denote the spectrum of $a \in A$: it is the image of the Gelfand transform of a, seen as a function $\mathcal{M}(A) \to \mathbb{C}$. Then the lifting problem has solution if and only if $\sigma(a) \subset U$.

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Axiomatic holomorphic functional calculus: (Rough) An algebra A together with subsets $A(U) \subset A^n$ for every $U \subset \mathbb{C}^n$ and (composable) holomorphic operations $f_A \colon A(U) \to A$ for every $f \colon U \to \mathbb{C}$.

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Reformulation: an axiomatic holomorphic functional calculus on a commutative $\mathbb{C}$-algebra A is a product-preserving functor

$$\mathcal{O} \colon \mathcal{T}_{\mathrm{an}}(\mathbb{C}) \to \mathrm{Set}$$

such that $\mathcal{O}(\mathbb{C}) = A$.

The k-analytic analogue of $\mathcal{T}_{\mathrm{an}}(\mathbb{C})$ is the following:

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1. products;
2. pullbacks along open immersions (in the $\mathbb{C}$-analytic case) or quasi-étale maps (in the k-analytic case).

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To any analytically ringed topos $(\mathcal{X}, \mathcal{O})$, we can always associate a simplicially ringed topos $(\mathcal{X}, \mathcal{O}^{\mathrm{alg}})$.

Derived analytic spaces

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Theorem (Lurie, P. - Yu)

1. *There exists an ∞-category $\mathrm{dAn}_\mathbb{C}$ (resp. dAn_k) of derived $\mathbb{C}$-analytic (resp. k-analytic) spaces.*
2. $\mathrm{dAn}_\mathbb{C}$ *and* dAn_k *admit fiber products.*
3. *There are fully faithful embeddings* $\mathrm{An}_\mathbb{C} \to \mathrm{dAn}_\mathbb{C}$ *and* $\mathrm{An}_k \to \mathrm{dAn}_k$*.*
4. *If $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a derived analytic space, then so is $(\mathcal{X}, \tau_{\leq n}\mathcal{O}_\mathcal{X})$ for every $n \geq 0$.*

Lurie's representability theorem

Theorem (P. - Yu)

Let $F \colon \mathrm{dStn}_{\mathbb{C}}^{\mathrm{op}} \to \mathcal{S}$ *be a sheaf for* τ_{an}*. The following conditions are equivalent:*

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   - (iii) *F is infinitesimally cartesian;*
   - (iv) *F admits a global analytic cotangent complex.*

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Example: let $\mathrm{B}^n(\mathbb{G}_a)$ be the n-th Eilenberg-Maclane stack associated to $\mathbb{G}_a$. Let $p\colon \mathrm{B}^n(\mathbb{G}_a) \to \mathrm{Spec}(k)$ be the natural map. Then

$$\mathrm{R}p_*(\mathcal{O}_{\mathrm{B}^n(\mathbb{G}_a)}) \simeq \mathrm{Sym}_k(k[-n]).$$

Question: what is the main source of nonconnective cdgas?

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Idea: enlarge $\mathcal{T}_{\mathrm{an}}(k)$ by adding the higher classifying stacks $\mathrm{B}^n(\mathbf{G}_a)$: let $\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)$ be the smallest full subcategory of analytic Artin stacks closed under products and containing both $\mathcal{T}_{\mathrm{an}}(k)$ and the collection $\{\mathrm{B}^n(\mathbf{G}_a)\}$.

Definition

Let $\mathcal{X}$ be an ∞-topos. A $\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)$-nonconnective structure is a functor

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3. for every $n \geq 0$ the diagram

$$\begin{array}{ccc} \mathcal{O}(\mathrm{B}^n(\mathbf{G}_a)) & \longrightarrow & \mathcal{O}(\mathsf{Spec}(k)) \\ \downarrow & & \downarrow \\ \mathcal{O}(\mathsf{Spec}(k)) & \longrightarrow & \mathcal{O}(\mathrm{B}^{n+1}(\mathbf{G}_a)) \end{array}$$

   is a pullback.

Remark

The collection $\{\mathcal{O}(\mathrm{B}^n(\mathbf{G}_a))\}_{n \geq 0}$ forms an Ω-spectrum in $\mathcal{X}$. This gives an *underlying spectrum* functor

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X}) \longrightarrow \mathsf{Sp}(X).$$

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Theorem (António, Petit, P.)

The connective cover functor

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induced by precomposition with $\mathcal{T}_{\mathrm{an}}(k) \hookrightarrow \mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)$ has a fully faithful left adjoint

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Thanks for the attention!