Grothendieck categories as a bilocalization of linear sites

Toposes in Como

Julia Ramos González

June 27, 2018

University of Antwerp

Motivation

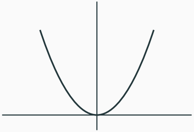
Algebra

Geometry

Algebra

$A = k[x, y]/\langle y - x^2 \rangle$

Geometry

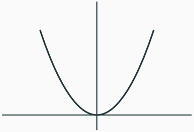


Noncommutative Algebraic Geometry

Algebra

$A = k[x, y]/\langle y - x^2 \rangle$

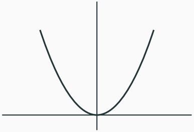
Geometry



Category Theory

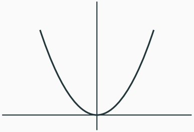
$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y-x^2 \rangle}\right)$

Noncommutative Algebraic Geometry

Algebra	Geometry	Category Theory
$A = k[x, y]/\langle y - x^2 \rangle$		$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y-x^2 \rangle}\right)$

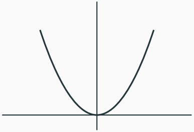
Noncommutative multiplication

Noncommutative Algebraic Geometry

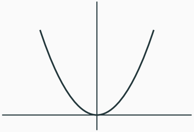
Algebra	Geometry	Category Theory
$A = k[x, y]/\langle y - x^2 \rangle$		$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y-x^2 \rangle}\right)$

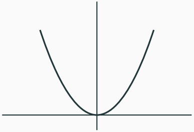
Noncommutative multiplication

?????

Algebra	Geometry	Category Theory
$A = k[x, y]/\langle y - x^2 \rangle$		$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y - x^2 \rangle}\right)$
Noncommutative multiplication	?????	Categorical information is still available!

Noncommutative Algebraic Geometry

Algebra	Geometry	Category Theory
$A = k[x,y]/\langle y - x^2 \rangle$		$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y - x^2 \rangle}\right)$
Noncommutative multiplication	?????	Categorical information is still available!
A NC ring	$\rightarrow$	$\mathrm{Mod}(A)$ - affine NC space

Algebra	Geometry	Category Theory
$A = k[x,y]/\langle y - x^2 \rangle$		$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y - x^2 \rangle}\right)$
Noncommutative multiplication	?????	Categorical information is still available!
A NC ring	$\rightarrow$	Mod(A) - affine NC space
A NC graded ring + ...	$\rightarrow$	Qgr(A) - projective NC space (Serre's thm)

Grothendieck categories

Which categories?

Grothendieck categories will be our models for NC spaces

Grothendieck categories

k – commutative ring

Definition

A *k-linear Grothendieck category* is a cocomplete k-linear abelian category with a generator and exact filtered colimits.

Which categories?

Grothendieck categories will be our models for NC spaces

k – commutative ring

Definition

A *k-linear Grothendieck category* is a cocomplete k-linear abelian category with a generator and exact filtered colimits.

Which categories?

Grothendieck categories will be our models for NC spaces

Motivation

Theorem (Gabber)

Let X be a scheme. $\mathrm{Qch}(X)$ *is a Grothendieck abelian category.*

Theorem (Gabriel-Rosenberg)

We can recover the geometry of a scheme X from $\mathrm{Qch}(X)$.

Linear sites and Grothendieck categories: the objects

Linear sites

$\mathfrak{a}$ – small k-linear category $\equiv$ enriched over $\mathrm{Mod}(k)$

Linear sites

$\mathfrak{a}$ – small k-linear category $\equiv$ enriched over $\mathsf{Mod}(k)$

Definition

A *k-linear Grothendieck topology* $\mathcal{T}$ on $\mathfrak{a}$ is a $\mathsf{Mod}(k)$-enriched version of the classical notion of Grothendieck topology, i.e. for every $A \in \mathfrak{a}$ the covering sieves are submodules $R \subseteq \mathfrak{a}(-, A)$ fulfilling the usual axioms of a Grothendieck topology in the enriched setup.

Linear sites

$\mathfrak{a}$ – small k-linear category $\equiv$ enriched over $\mathrm{Mod}(k)$

Definition

A *k-linear Grothendieck topology* $\mathcal{T}$ on $\mathfrak{a}$ is a $\mathrm{Mod}(k)$-enriched version of the classical notion of Grothendieck topology, i.e. for every $A \in \mathfrak{a}$ the covering sieves are submodules $R \subseteq \mathfrak{a}(-, A)$ fulfilling the usual axioms of a Grothendieck topology in the enriched setup.

Definition

A *k-linear site* is a pair $(\mathfrak{a}, \mathcal{T})$ as above.

Linear sites

$\mathfrak{a}$ – small k-linear category $\equiv$ enriched over $\mathsf{Mod}(k)$

Definition

A *k-linear Grothendieck topology* $\mathcal{T}$ on $\mathfrak{a}$ is a $\mathsf{Mod}(k)$-enriched version of the classical notion of Grothendieck topology, i.e. for every $A \in \mathfrak{a}$ the covering sieves are submodules $R \subseteq \mathfrak{a}(-, A)$ fulfilling the usual axioms of a Grothendieck topology in the enriched setup.

Definition

A *k-linear site* is a pair $(\mathfrak{a}, \mathcal{T})$ as above.

This is one instance of the enriched sheaf theory introduced by Borceux and Quinteiro in [BQ96].

This particular example has been analysed later on by Lowen in [Low16] with deformation theory purposes.

Linear sheaves and presheaves

$(\mathfrak{a}, \mathcal{T})$ – k-linear site

Linear sheaves and presheaves

$(\mathfrak{a}, \mathcal{T})$ – k-linear site

Definition

- *k-linear presheaves*: $\mathrm{Mod}(\mathfrak{a}) := \mathrm{Fun}_k(\mathfrak{a}^{\mathrm{op}}, \mathrm{Mod}(k))$
- *k-linear sheaves*: $\mathrm{Sh}(\mathfrak{a}, \mathcal{T}) \subseteq \mathrm{Mod}(\mathfrak{a})$ the full subcategory of k-linear presheaves F such that the restriction

$$Mod(\mathfrak{a})(\mathfrak{a}(-, A), F) \xrightarrow{\cong} \mathrm{Mod}(\mathfrak{a})(R, F),$$

for all $A \in \mathfrak{a}$ and all $R \in \mathcal{T}(A)$.

Linear sheaves and presheaves

$(\mathfrak{a}, \mathcal{T})$ – k-linear site

Definition

- *k-linear presheaves*: $\mathrm{Mod}(\mathfrak{a}) := \mathrm{Fun}_k(\mathfrak{a}^{\mathrm{op}}, \mathrm{Mod}(k))$
- *k-linear sheaves*: $\mathrm{Sh}(\mathfrak{a}, \mathcal{T}) \subseteq \mathrm{Mod}(\mathfrak{a})$ the full subcategory of k-linear presheaves F such that the restriction

$$Mod(\mathfrak{a})(\mathfrak{a}(-, A), F) \xrightarrow{\cong} \mathrm{Mod}(\mathfrak{a})(R, F),$$

for all $A \in \mathfrak{a}$ and all $R \in \mathcal{T}(A)$.

Proposition (Borceux-Quinteiro)

The inclusion $\mathrm{Sh}(\mathfrak{a}, \mathcal{T}) \subseteq \mathrm{Mod}(\mathfrak{a})$ *is a localization functor. Its k-linear exact left adjoint* $\# : \mathrm{Mod}(\mathfrak{a}) \longrightarrow \mathrm{Sh}(\mathfrak{a}, \mathcal{T})$ *is called* sheafification.

The Gabriel-Popescu theorem

Theorem (Gabriel-Popescu, generalization by Lowen)

k-Linear Grothendieck categories are the categories of sheaves over k-linear sites (k-linear Giraud theorem).

The Gabriel-Popescu theorem

Theorem (Gabriel-Popescu, generalization by Lowen)
k-Linear Grothendieck categories are the categories of sheaves over k-linear sites (k-linear Giraud theorem).

Remark
For each Grothendieck category $\mathcal{C}$, there exist multiple choices of linear sites $(\mathfrak{a}, \mathcal{T})$ such that $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$.

Grothendieck categories and linear sites: the (1-)categories

The (1)-categories of Grothendieck categories

For **geometric** interests:

- The (1-)category Grt with:
  - $\mathrm{Obj}(\mathsf{Grt}) = \{k\text{-linear Grothendieck categories}\}$
  - $\mathsf{Grt}(\mathcal{A}, \mathcal{B}) = \{\text{colimit preserving } k\text{-linear functors } \mathcal{A} \to \mathcal{B}\}$

The (1)-categories of Grothendieck categories

For **geometric** interests:

- The (1-)category Grt with:
  - $\mathrm{Obj}(\mathrm{Grt}) = \{k\text{-linear Grothendieck categories}\}$
  - $\mathrm{Grt}(\mathcal{A}, \mathcal{B}) = \{\text{colimit preserving } k\text{-linear functors } \mathcal{A} \to \mathcal{B}\}$

For **classical topos theory** interests:

- The (1-)category Topos_k with:
  - $\mathrm{Obj}(\mathrm{Topos}_k) = \{k\text{-linear Grothendieck categories}\}$
  - $\mathrm{Topos}_k(\mathcal{A}, \mathcal{B}) = \{\text{geometric } k\text{-linear functors } F^* : \mathcal{B} \rightleftarrows \mathcal{A} : F_*\}$

The (1-)categories of linear sites

For **geometric** interests:

- The (1-)category $\mathrm{Site}_{k,\mathrm{cont}}$ with:
  - $\mathrm{Obj}(\mathrm{Site}_{k,\mathrm{cont}}) = \{k\text{-linear sites}\}$
  - $\mathrm{Site}_{k,\mathrm{cont}}((\mathfrak{a}, \mathcal{T}_\mathfrak{a}), (\mathfrak{b}, \mathcal{T}_\mathfrak{b})) =$
    $\{\text{continuous } k\text{-linear functors } f : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \to (\mathfrak{b}, \mathcal{T}_\mathfrak{b})\} =$
    $\{k\text{-linear } f : \mathfrak{a} \to \mathfrak{b} \mid f^* : \mathrm{Mod}(\mathfrak{b}) \to \mathrm{Mod}(\mathfrak{a}) : M \mapsto M \circ f$
    restricts to a map $f_s : \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b}) \to \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a})\}$

The (1-)categories of linear sites

For **geometric** interests:

- The (1-)category $\mathsf{Site}_{k,\mathrm{cont}}$ with:
  - $\mathrm{Obj}(\mathsf{Site}_{k,\mathrm{cont}}) = \{k\text{-linear sites}\}$
  - $\mathsf{Site}_{k,\mathrm{cont}}((\mathfrak{a}, \mathcal{T}_\mathfrak{a}), (\mathfrak{b}, \mathcal{T}_\mathfrak{b})) =$
    $\{\text{continuous } k\text{-linear functors } f : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \to (\mathfrak{b}, \mathcal{T}_\mathfrak{b})\} =$
    $\{k\text{-linear } f : \mathfrak{a} \to \mathfrak{b} \mid f^* : \mathsf{Mod}(\mathfrak{b}) \to \mathsf{Mod}(\mathfrak{a}) : M \mapsto M \circ f$
    restricts to a map $f_s : \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b}) \to \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a})\}$

Recall:

Proposition (SGA 4)

If $f : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \longrightarrow (\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ is a continuous morphism, there exists a functor $f^s : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \longrightarrow \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ such that $f^s \dashv f_s$, and in particular, it is colimit preserving.

The (1-)categories of linear sites

For **geometric** interests:

- The (1-)category $\mathsf{Site}_{k,\mathrm{cont}}$ with:
  - $\mathrm{Obj}(\mathsf{Site}_{k,\mathrm{cont}}) = \{k\text{-linear sites}\}$
  - $\mathsf{Site}_{k,\mathrm{cont}}((\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}), (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})) =$
    $\{\text{continuous } k\text{-linear functors } f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})\} =$
    $\{k\text{-linear } f : \mathfrak{a} \to \mathfrak{b} \mid f^* : \mathsf{Mod}(\mathfrak{b}) \to \mathsf{Mod}(\mathfrak{a}) : M \mapsto M \circ f$
    $\text{restricts to a map } f_s : \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \to \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})\}$

For **classical topos theory** interests:

- The (1-)category Site_{k} with:
  - $\mathrm{Obj}(\mathsf{Site}_{k}) = \{k\text{-linear sites}\}$
  - $\mathsf{Site}_{k}((\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}), (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})) =$
    $\{k\text{-linear morphisms of sites } f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})\} =$
    $\{\text{continuous } k\text{-linear functors } f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \text{ such that}$
    $f^s : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \text{ is exact}\}$

From linear sites to Grothendieck categories

We can define (pseudo)functors:

- $\phi : \mathsf{Site}_{k,\mathrm{cont}} \longrightarrow \mathsf{Grt}$ given by:

$$\phi(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) := \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$$

$$\phi\,[f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})] := [f^{s} : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})]$$

From linear sites to Grothendieck categories

We can define (pseudo)functors:

- $\phi : \mathsf{Site}_{k,\mathrm{cont}} \longrightarrow \mathsf{Grt}$ given by:

$$\phi(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) := \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$$

$$\phi\,[f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})] := [f^s : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})]$$

- $\psi : \mathsf{Site}_k \longrightarrow \mathsf{Topos}_k^{\mathrm{op}}$ given by:

$$\psi(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) := \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$$

$$\psi\,[f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})] := [f^s : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightleftarrows \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) : f_s]$$

The localization intuition

Lemme de comparaison (LC) morphisms

Theorem (SGA 4, Lowen, ...)

Let $f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \longrightarrow (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ *be a continuous* k*-linear morphism such that* $f^{-1}\mathcal{T}_{\mathfrak{b}} = \mathcal{T}_{\mathfrak{a}}$ *and satisfying:*

Theorem (SGA 4, Lowen, ...)

Let $f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \longrightarrow (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ be a continuous k-linear morphism such that $f^{-1}\mathcal{T}_{\mathfrak{b}} = \mathcal{T}_{\mathfrak{a}}$ and satisfying:

(G) $\{u_A : f(A) \to B\}_{u_A}$ *generates a covering of B in $(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$*

Lemme de comparaison (LC) morphisms

Theorem (SGA 4, Lowen, ...)

Let $f : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \longrightarrow (\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ *be a continuous k-linear morphism such that* $f^{-1}\mathcal{T}_\mathfrak{b} = \mathcal{T}_\mathfrak{a}$ *and satisfying:*

(G) $\{u_A : f(A) \to B\}_{u_A}$ *generates a covering of* B *in* $(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$

(F) f *is "full up to covers in* $(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$*"*

Lemme de comparaison (LC) morphisms

Theorem (SGA 4, Lowen, ...)

Let $f : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \longrightarrow (\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ *be a continuous k-linear morphism such that* $f^{-1}\mathcal{T}_\mathfrak{b} = \mathcal{T}_\mathfrak{a}$ *and satisfying:*

(G) $\{u_A : f(A) \to B\}_{u_A}$ *generates a covering of B in* $(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$

(F) *f is "full up to covers in* $(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$*"*

(FF) *f is "faithful up to covers in* $(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$*"*

Lemme de comparaison (LC) morphisms

Theorem (SGA 4, Lowen, ...)

Let $f : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \longrightarrow (\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ be a continuous k-linear morphism such that $f^{-1}\mathcal{T}_\mathfrak{b} = \mathcal{T}_\mathfrak{a}$ and satisfying:

(G) $\{u_A : f(A) \to B\}_{u_A}$ *generates a covering of B in $(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$*

(F) *f is "full up to covers in $(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$"*

(FF) *f is "faithful up to covers in $(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$"*

Then $f_s : \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b}) \to \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$ is an equivalence, and hence so is f^s. A continuous f with those properties is called an LC morphism.

Lemme de comparaison (LC) morphisms

Theorem (SGA 4, Lowen, ...)

Let $f : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \longrightarrow (\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ *be a continuous* k*-linear morphism such that* $f^{-1}\mathcal{T}_\mathfrak{b} = \mathcal{T}_\mathfrak{a}$ *and satisfying:*

(G) $\{u_A : f(A) \to B\}_{u_A}$ *generates a covering of* B *in* $(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$

(F) f *is "full up to covers in* $(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$*"*

(FF) f *is "faithful up to covers in* $(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$*"*

Then $f_s : \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b}) \to \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$ *is an equivalence, and hence so is* f^s*. A continuous* f *with those properties is called an LC morphism.*

Remark

Observe that every LC morphism is in particular a k*-linear morphism of sites, hence we have:*

$$\mathrm{LC}((\mathfrak{a}, \mathcal{T}_\mathfrak{a}), (\mathfrak{b}, \mathcal{T}_\mathfrak{b})) \subseteq \mathrm{Site}_k((\mathfrak{a}, \mathcal{T}_\mathfrak{a}), (\mathfrak{b}, \mathcal{T}_\mathfrak{b})) \subseteq \mathrm{Site}_{k,\mathrm{cont}}((\mathfrak{a}, \mathcal{T}_\mathfrak{a}), (\mathfrak{b}, \mathcal{T}_\mathfrak{b}))$$

The roof theorem

Theorem (Stacks Project + RG)

Given $F : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \to \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ *colimit preserving, there exists a subcanonical site* $(\mathfrak{c}, \mathcal{T}_\mathfrak{c})$ *and continuous morphisms* f, u *as in*

$$\begin{array}{ccccc} & & (\mathfrak{c}, \mathcal{T}_\mathfrak{c}) & & \\ & \overset{f}{\nearrow} & & \overset{u}{\nwarrow} & \\ (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) & & & & (\mathfrak{b}, \mathcal{T}_\mathfrak{b}), \end{array}$$

with u *an LC morphism, such that* $F = u_s \circ f^s$.

Theorem (Stacks Project + RG)

Given $F : \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ *colimit preserving, there exists a subcanonical site* $(\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ *and continuous morphisms* f, u *as in*

$$\begin{array}{ccccc} & & (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}}) & & \\ & \overset{f}{\nearrow} & & \overset{u}{\nwarrow} & \\ (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) & & & & (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}), \end{array}$$

with u *an LC morphism, such that* $F = u_s \circ f^s$.

GABRIEL-ZISMAN LOCALIZATION: Morphisms in Grt are obtained inverting LC morphisms in $\mathrm{Site}_{k,\mathrm{cont}}$.

Theorem (Stacks Project + RG)

Given $F^* : \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \rightleftarrows \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b}) : F_*$ geometric morphism, there exists a subcanonical site $(\mathfrak{c}, \mathcal{T}_\mathfrak{c})$ and morphisms of sites f, u as in

$$(\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \xrightarrow{\ f\ } (\mathfrak{c}, \mathcal{T}_\mathfrak{c}) \xleftarrow{\ u\ } (\mathfrak{b}, \mathcal{T}_\mathfrak{b}),$$

with u an LC morphism, such that $F^* = u_s \circ f^s$ and $F_* = f_s \circ u^s$.

The roof theorem

Theorem (Stacks Project + RG)

Given $F^* : \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \rightleftarrows \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b}) : F_*$ *geometric morphism, there exists a subcanonical site* $(\mathfrak{c}, \mathcal{T}_\mathfrak{c})$ *and morphisms of sites* f, u *as in*

$$
\begin{array}{ccccc}
 & & (\mathfrak{c}, \mathcal{T}_\mathfrak{c}) & & \\
 & \overset{f}{\nearrow} & & \overset{u}{\nwarrow} & \\
(\mathfrak{a}, \mathcal{T}_\mathfrak{a}) & & & & (\mathfrak{b}, \mathcal{T}_\mathfrak{b}),
\end{array}
$$

with u *an LC morphism, such that* $F^* = u_s \circ f^s$ *and* $F_* = f_s \circ u^s$.

GABRIEL-ZISMAN LOCALIZATION: Morphisms in $\mathrm{Topos}_k^{\mathrm{op}}$ are obtained inverting LC morphisms in Site_k.

Localization in the bicategorical setup

The 2-categories

For **geometric** interests, we consider:

- The 2-category Grt as before, with 2-morphisms given by the k-linear natural transformations $A : F \Rightarrow G$ between colimit preserving functors
- The 2-category $\mathrm{Site}_{k,\mathrm{cont}}$ as before, with 2-morphisms given by the k-linear natural trasnformations $\alpha : f \Rightarrow g$ between continous functors

The 2-categories

For **geometric** interests, we consider:

- The 2-category Grt as before, with 2-morphisms given by the k-linear natural transformations $A : F \Rightarrow G$ between colimit preserving functors
- The 2-category $\text{Site}_{k,\text{cont}}$ as before, with 2-morphisms given by the k-linear natural trasnformations $\alpha : f \Rightarrow g$ between continous functors

For **classical topos theory** interests, we consider:

- The 2-category Topos_k as before, with 2-morphisms given by the k-linear natural transformations $A : F_* \Rightarrow G_*$ between the right adjoints of the geometric morphisms
- The 2-category Site_k as before, with 2-morphisms given by the k-linear natural trasnformations $\alpha : f \Rightarrow g$ between morphisms of sites

Pronk's bicategories of fractions

Pronk introduces in [Pro96] a suitable generalization to the bicategorical setting of the 1-categorical notion of a class of (1-)morphisms admitting a left/right calculus of fractions and defines bicategories of fractions in this setup.

Pronk's bicategories of fractions

Pronk introduces in [Pro96] a suitable generalization to the bicategorical setting of the 1-categorical notion of a class of (1-)morphisms admitting a left/right calculus of fractions and defines bicategories of fractions in this setup.

Definition (Pronk)

Let $\mathcal{C}$ be a bicategory and W a class of 1-morphisms admitting a left calculus of fractions. A *bilocalization of* $\mathcal{C}$ *along* W is a pair $(\mathcal{C}[\mathsf{W}^{-1}], \Lambda)$ of a bicategory and a pseudofunctor such that:

1. $\Lambda : \mathcal{C} \to \mathcal{C}[\mathsf{W}^{-1}]$ sends elements in W to equivalences;
2. Composition with Λ gives an equivalence of bicategories

$$\mathrm{Hom}(\mathcal{C}[\mathsf{W}^{-1}], \mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathsf{W}}(\mathcal{C}, \mathcal{D})$$

for each bicategory $\mathcal{D}$.

Main result

Proposition

LC *admits a left calculus of fractions in* $\mathrm{Site}_{k,\mathrm{cont}}$.

Main result

Proposition

LC *admits a left calculus of fractions in* $\mathsf{Site}_{k,\mathrm{cont}}$.

Theorem ([Ram18])

There exists a pseudofunctor

$$\Phi : \mathsf{Site}_{k,\mathrm{cont}} \to \mathsf{Grt}$$

which sends LC morphisms to equivalences in Grt, *such that the pseudofunctor*

$$\tilde{\Phi} : \mathsf{Site}_{k,\mathrm{cont}}[\mathsf{LC}^{-1}] \to \mathsf{Grt}$$

induced by Φ *via the universal property of the bilocalization is an equivalence of bicategories.*

Main result

Proposition

LC *admits a left calculus of fractions in* Site_k.

Theorem ([Ram18])

There exists a pseudofunctor

$$\Psi : \mathrm{Site}_k \to \mathrm{Topos}_k^{\mathrm{coop}}$$

which sends LC morphisms to equivalences in $\mathrm{Topos}_k^{\mathrm{coop}}$, *such that the pseudofunctor*

$$\tilde{\Psi} : \mathrm{Site}_k[\mathrm{LC}^{-1}] \to \mathrm{Topos}_k^{\mathrm{coop}}$$

induced by Ψ *via the universal property of the bilocalization is an equivalence of bicategories.*

Sketch of the proof

1. **Extend** $\phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 1**-categories to a pseudofunctor** $\Phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 2**-categories**:

For $f, g : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ continuous functors,

$$\Phi(\alpha : f \Rightarrow g) := \alpha^s : f^s \Rightarrow g^s$$

defined by adjunction from $\alpha_s : g_s \Rightarrow f_s$, where

$$(\alpha_s)_F(A) := F(\alpha_A) : g_s(F)(A) = F(g(A)) \to F(f(A)) = f_s(F)(A),$$

for all $F \in \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$, all $A \in \mathfrak{a}$.

Sketch of the proof

1. **Extend** $\phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 1**-categories to a pseudofunctor** $\Phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 2**-categories**:

For $f, g : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ continuous functors,

$$\Phi(\alpha : f \Rightarrow g) := \alpha^s : f^s \Rightarrow g^s$$

defined by adjunction from $\alpha_s : g_s \Rightarrow f_s$, where

$$(\alpha_s)_F(A) := F(\alpha_A) : g_s(F)(A) = F(g(A)) \to F(f(A)) = f_s(F)(A),$$

for all $F \in \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$, all $A \in \mathfrak{a}$.

2. **Observe** Φ **sends LC morphisms to equivalences**.

Sketch of the proof

1. **Extend** $\phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 1**-categories to a pseudofunctor** $\Phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 2**-categories**:

For $f, g : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ continuous functors,

$$\Phi(\alpha : f \Rightarrow g) \coloneqq \alpha^s : f^s \Rightarrow g^s$$

defined by adjunction from $\alpha_s : g_s \Rightarrow f_s$, where

$$(\alpha_s)_F(A) \coloneqq F(\alpha_A) : g_s(F)(A) = F(g(A)) \to F(f(A)) = f_s(F)(A),$$

for all $F \in \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$, all $A \in \mathfrak{a}$.

2. **Observe** Φ **sends LC morphisms to equivalences**.
3. **One proves that** Φ **satisfies Tommasini's criterion.**

Sketch of the proof

Tommasini's criterion [Tom]: Provides necessary and sufficient conditions for $A : \mathcal{C} \to \mathcal{D}$ sending a class W of 1-morphisms with a left calculus of fractions in $\mathcal{C}$ to equivalences in $\mathcal{D}$, so that its induced pseudofunctor $\tilde{A} : \mathcal{C}[\mathsf{W}^{-1}] \to \mathcal{D}$ is an equivalence of bicategories.

Sketch of the proof

1. **Extend** $\phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 1**-categories to a pseudofunctor** $\Phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 2**-categories**:

For $f, g : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ continuous functors,

$$\Phi(\alpha : f \Rightarrow g) := \alpha^s : f^s \Rightarrow g^s$$

defined by adjunction from $\alpha_s : g_s \Rightarrow f_s$, where

$$(\alpha_s)_F(A) := F(\alpha_A) : g_s(F)(A) = F(g(A)) \to F(f(A)) = f_s(F)(A),$$

for all $F \in \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$, all $A \in \mathfrak{a}$.

2. **Observe** Φ **sends LC morphisms to equivalences**.
3. **One proves that** Φ **satisfies Tommasini's criterion.**

Sketch of the proof

1. Extend $\psi : \mathrm{Site}_k \to \mathrm{Topos}_k^{\mathrm{op}}$ between the 1-categories to a pseudofunctor $\Psi : \mathrm{Site}_k \to \mathrm{Topos}_k^{\mathrm{coop}}$ between the 2-categories:

For $f, g : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ morphisms of sites,

$$\Psi(\alpha : f \Rightarrow g) := \alpha_s : g_s \Rightarrow f_s$$

defined as

$$(\alpha_s)_F(A) := F(\alpha_A) : g_s(F)(A) = F(g(A)) \to F(f(A)) = f_s(F)(A),$$

for all $F \in \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$, all $A \in \mathfrak{a}$.

2. Observe Ψ sends LC morphisms to equivalences.
3. One proves that Ψ satisfies Tommasini's criterion.

Remarks

- The analogous results can be obtained analogously in the set-theoretical setup.

Remarks

- The analogous results can be obtained analogously in the set-theoretical setup.
- We can safely define constructions in the 2-category of Grothendieck categories at the level of the sites as long as they behave well with respect to LC morphisms

One instance: Monoidal structure

Our original motivation: Study of a monoidal structure in Grt

- In [LRS17] we have defined a tensor product of Grothendieck categories based on a tensor product of linear sites, which defines a monoidal structure in $\mathsf{Site}_{k,\mathrm{cont}}$.

One instance: Monoidal structure

Our original motivation: Study of a monoidal structure in Grt

- In [LRS17] we have defined a tensor product of Grothendieck categories based on a tensor product of linear sites, which defines a monoidal structure in $\mathsf{Site}_{k,\mathrm{cont}}$.
- In addition, it is also shown in [LRS17] that LC morphisms are closed with respect to this tensor product of sites.

One instance: Monoidal structure

Our original motivation: Study of a monoidal structure in Grt

- In [LRS17] we have defined a tensor product of Grothendieck categories based on a tensor product of linear sites, which defines a monoidal structure in $\mathrm{Site}_{k,\mathrm{cont}}$.
- In addition, it is also shown in [LRS17] that LC morphisms are closed with respect to this tensor product of sites.
- A bicategorical version of monoidal localization à la Day [Day73], developed in unpublished work by Pronk, would immediately provide us with a monoidal structure in Grt, where the tensor product "is given" by the one defined in [LRS17].

Thank you for your attention

References

F. Borceux and C. Quinteiro. **A theory of enriched sheaves.** *Cah. Topol. Géom. Différ. Catég.*, 37(2):145–162, 1996.

B. Day. **Note on monoidal localisation.** *Bull. Austral. Math. Soc.*, (8):1–16, 1973.

W. Lowen. **Linearized topologies and deformation theory.** *Topology Appl.*, 200:176–211, 2016.

W. Lowen, J. Ramos González, and B. Shoikhet. **On the tensor product of linear sites and Grothendieck categories.** *Int. Math. Res. Not. IMRN*, 2017.

D. Pronk. **Etendues and stacks as bicategories of fractions.** *Compositio Math.*, (3):24–303, 1996.

J. Ramos González. **Grothendieck categories as a bilocalization of linear sites.** *Appl. Categor. Struct.*, 2018.

M. Tommasini. **Some insights on bicategories of fractions III - Equivalences of bicategories of fractions.** arXiv:1410.6395 [math.CT].