TOPOSES IN COMO 2018

Eduardo J. Dubuc

$$\begin{tikzcd}
 & & F \arrow[dll, bend right, "x"'] \arrow[dl, "a"'] \arrow[d, "z"] \arrow[dr, "s"] \arrow[ddrr, bend left, "y"] & & \\
 & A \arrow[dl, "\pi_x"'] & Z \arrow[dll] \arrow[drr] & A_0 & \\
X & & & & Y
\end{tikzcd}$$

$$\mathcal{C} \xrightarrow{F} \mathcal{S}$$

$$\Gamma_F : (x, X) \mid F \xrightarrow{x} X$$
$$\equiv$$
$$x \in FX$$

$$\Gamma_F \subset \left(\mathcal{S}^{\mathcal{C}}\right)^{op}$$

↑ Yoneda

$$A = \bigwedge_{x \in FX} X$$

↳ Galois Closure

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GALOIS THEORY

$\boxed{\text{Theorem} \quad \hat{F} \text{ is an equivalence}}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[drr, "F"'] \arrow[rrrr, bend left, "\hat{F}"] & \mathcal{E} \arrow[dr] \arrow[rr] & & \beta^G \arrow[dl] \arrow[r, leftrightarrow] & \beta_t^G \arrow[dll] \\
& & \mathcal{S} & &
\end{tikzcd}$$

F = fiber functor

$\mathcal{C}$ site of connected objects

$\mathcal{E}$ atomic topos (connected)

F inverse image of a point

(recall F preserves finite limits)

G automorphism group of F

GALOIS-GALOIS

F preserves all limits

$\equiv$ F is representable $F = [A_0, -]$

$G = Aut(A_0)^{op}$, G discrete.

SGA 1

F finite set valued

$\Rightarrow$ $\exists$ists Galois Closure

$G = \varprojlim_{(a,A)} Aut(A)^{op}$ $Aut(A)$ finite

G profinite.

SGA 4

Galois Topos

F preserves cotensors

$\Rightarrow$ $\exists$ists Galois Closure

G is strict progroup $\{Aut(A)\}^{op}_{(a,A)}$

$\equiv$ $G = \varprojlim_{(a,A)} Aut(A)^{op}$ in localic groups

G prodiscrete localic

LOCALIC GALOIS

Atomic Topos

F any point

G = localic group of automorphisms

G is a localic group

TANNAKA THEORY (general)

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\hat{T}"] \arrow[dr, "T"'] & & \mathrm{Com}_0(H) \arrow[dl] \\ & \mathcal{V}_0 \hookrightarrow \mathcal{V} & \end{tikzcd}$$

T = fiber functor

$\mathcal{V}$ cocomplete monoidal closed

$\mathcal{V}_0$ objects with right dual

H = coalgebra of endomorphisms of T

Nat predual (Joyal terminology)

$$\mathrm{Nat}^{\vee}(L,T) = \int^X \underbrace{LX \otimes (TX)^{\wedge}}_{\hom(LX,TX)^{\wedge}}$$

it follows $\mathrm{Nat}(L,T) = \mathrm{Nat}^{\vee}(L,T)^*$

$$\boxed{H^* = \mathcal{V}(H,2) = \mathrm{Nat}(T)}$$

$$H = \mathrm{End}^{\vee}(T) = \mathrm{Nat}^{\vee}(T,T)$$

i) H is a coalgebra, and there is $TX \longrightarrow \mathrm{End}^{\vee}(T) \otimes TX$ comodule structure

ii) $\mathcal{A}$ monoidal, T monoidal $\Rightarrow$ H bialgebra

iii) $\mathcal{A}$ symmetric $\Rightarrow$ H is commutative algebra

iv) All objects of $\mathcal{A}$ have duals $\Rightarrow$ H is a Hopf algebra

Alternative Terminology $\mathcal{A}$ is rigid symmetric tensor

Example $\mathcal{V} = \mathrm{Vec}_K$ $\mathcal{V}_0 = \mathcal{V}_K^{<\infty}$

$\boxed{\text{Theorem}}$ $\mathcal{A}$ abelian, T faithful $\Rightarrow$ $\hat{T}$ equivalence

TERMINOLOGY

sl monoidal category of sup lattices with linear maps (sup preserving)

Locale. idempotent algebra in sl

Localic group group object in sl^{op}

Localic group $\equiv$ Idempotent Hopf algebra

$X \in \mathcal{S}$ **ℓX** power set with direct image is the free sup lattice on X

$$\frac{R \subset X \times Y}{\ell X \to \ell Y}$$
linear

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\ell"] \arrow[dr] & sl \\ & \mathcal{R}el \arrow[u, "\ell"'] \end{tikzcd}$$

$\mathcal{R}el$ = category of relations

Bijections with values in a locale

$X \times Y \xrightarrow{\lambda} G$

ed. $\bigvee_y \lambda(x,y) = 1$ each x

uv. $\lambda(x,y) \wedge \lambda(x,y') = 0$ $y \neq y'$

su. $\bigvee_x \lambda(x y) = 1$ each y

in. $\lambda(x,y) \wedge \lambda(x' y) = 0$ $x \neq x'$

ACTIONS VERSUS COMODULES

Localic group of automorphisms $=_{def}$ universal bijection

$$\begin{tikzcd} & \mathrm{Rel}(X) \arrow[dr] & \\ X\times X \arrow[ur, bend left] \arrow[rr, "\langle 1 \rangle"'] & & \mathrm{Aut}(X) \end{tikzcd}$$

$\langle x,y\rangle$ is the subbase of the product topology

Coalgebra structure $\quad \omega(x,y) = \bigvee_z \langle x|z\rangle \otimes \langle z|y\rangle$

$$\begin{tikzcd} X\times X \arrow[r] \arrow[dr, "\omega"'] & \mathrm{Rel}(X) \arrow[r] \arrow[d, "\omega"] & \mathrm{Aut}(X) \arrow[d, "\omega"] \\ & \mathrm{Rel}(X)\otimes \mathrm{Rel}(X) \arrow[r] & \mathrm{Aut}(X)\otimes \mathrm{Aut}(X) \end{tikzcd}$$

Aut(X) it is an idempotent Hopf algebra $\equiv$ localic group

$$\begin{tikzcd} X\times X \arrow[r, "\langle 1 \rangle"] \arrow[dr, "\langle 1 \rangle^{op}"'] & \mathrm{Rel}(X) \arrow[r] \arrow[d, "\iota"] & \mathrm{Aut}(X) \arrow[d, "\iota"] \\ & \mathrm{Rel}(X) \arrow[r] & \mathrm{Aut}(X) \end{tikzcd} \qquad \begin{tikzcd} X\times X \arrow[r] \arrow[dr, "\delta"'] & \mathrm{Rel}(X) \arrow[r] \arrow[d] & \mathrm{Aut}(X) \arrow[dl] \\ & 2 & \end{tikzcd}$$

Prop [M.G]

$$\begin{tikzcd} X\times X \arrow[r] \arrow[dr, "\mu"'] & \mathrm{Rel}(X) \arrow[d, "\mu"] \\ & G \end{tikzcd}$$

G localic group, μ coalgebra morphism $\Longrightarrow$ μ bijection

Action of G on X

Axioms of action

$$\begin{tikzcd} X\times X \arrow[r] \arrow[dr, "\mu"', "\text{bijection}"] & \mathrm{Aut}(X) \arrow[d, "\mu"] \\ & G \end{tikzcd}$$

Morphism of actions

$$\begin{tikzcd} X\times X \arrow[rr, "f\times f"] \arrow[dr, "\mu"'] & & X'\times X' \arrow[dl, "\mu'"] \\ & G & \end{tikzcd}$$

Comodules of G on ℓX

$$\ell X \xrightarrow{e} G\otimes \ell X$$

Axioms of comodule

Morphism of comodules as usual

$$G\ \text{action of } G \text{ on } X \quad \equiv \quad G \text{ comodule structure on } \ell X$$

<u>proof</u>

$$\frac{\dfrac{\dfrac{\mathcal{R}\mathrm{el}(X) \xrightarrow{\mu} G \quad \text{locale morphism}}{X \times X \xrightarrow{\mu} G \quad \text{relation}}}{\ell X \otimes \ell X \xrightarrow{\mu} G \quad \text{linear}}}{\ell X \xrightarrow{\rho} G \otimes \ell X \quad \text{linear}}$$

Can check $\quad \rho\ G$ comodule $\iff \mu$ coalgebra morphism

$$\Updownarrow \ \text{Prop}\ \boxed{MG}$$

$\mu\ G$ action $\blacksquare$

Also $\quad$ Relations between G sets $\quad$ correspond $\quad$ Morphisms of comodules

$$R \rightarrowtail X \times Y \qquad\qquad \ell X \longrightarrow \ell Y$$

<u>Theorem I</u> $\quad$ There is an isomorphism of sl-categories

$$\begin{tikzcd} \mathcal{R}\mathrm{el}(\beta^G) \arrow[rr, "\cong"] \arrow[dr] & & \mathrm{Com}_0(G) \arrow[dl] \\ & \mathcal{R}\mathrm{el} \xrightarrow[\ell]{\cong} sl_0 & \end{tikzcd}$$

$$\boxed{\frac{R \rightarrowtail X \times Y \text{ in } \beta^G}{\ell X \to \ell Y \ \text{comodule morphism}}}$$

$$\Updownarrow \ \text{lifts}$$

$$\boxed{\frac{R \rightarrowtail X \times Y}{\ell X \to \ell Y \ \text{linear}}}$$

LOCALIC GALOIS THEORY

Localic Group of Automorphisms of F

$\mathcal{E} \xrightarrow{F} \mathcal{S}$ is defined by a universal property in the category of Locales

$$
\forall \begin{array}{c} X \\ \downarrow f \\ X' \end{array}
\qquad
\begin{tikzcd}
FX \times FX \arrow[d, "Ff \times Ff"] \arrow[dr, "\langle 1 \rangle_X"] \arrow[drr, bend left, "\varphi_X"] & & \\
 & Aut(F) \arrow[r, dashed, "\varphi"] & L \\
FX' \times FX' \arrow[ur, "\langle 1 \rangle_{X'}"'] \arrow[urr, bend right, "\varphi_{X'}"'] & &
\end{tikzcd}
$$

$\boxed{\rhd\text{-cone}}$

$\langle 1 \rangle_X$ φ_X bijections, $\exists!\, \varphi$ locale morphism.

$Aut(F) \longrightarrow 2$ locale morphism $\equiv$ Natural isomorphism of F

$FX \times FX \xrightarrow{\omega_X} Aut(F) \otimes Aut(F)$, $FX \times FX \xrightarrow{\iota_X} Aut(F)$, $FX \times FX \xrightarrow{\delta_X} 2$

Defined as before are $\rhd$-cones of bijections

Follows $Aut(F)$ is a localic group and $\langle 1 \rangle_X$ is an action of $Aut(F)$ on FX, and $F(f)$ is a morphism of actions.

There is a lifting
$$
\begin{tikzcd}
 & \beta^G \arrow[d] \\
\mathcal{E} \arrow[ur, "\hat{F}"] \arrow[r, "F"'] & \mathcal{S}
\end{tikzcd}
\qquad G = Aut(F).
$$

Theorem small $\mathcal{C} \hookrightarrow \mathcal{E}$ generators, $\rhd$-cone over $\mathcal{C}$ suffices $\Rightarrow$ $Aut(F)$ exists

Theorem $\mathcal{E}$ connected atomic, $\mathcal{C}$ conected generators $\Longrightarrow$ $\hat{F}$ equivalence

TANAKA CONTEXT FOR GALOIS

$$
\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "F"] & \mathcal{R}el(\mathcal{E}) \arrow[d, "T"] \\
\mathcal{S} \arrow[r] & \mathcal{R}el \underset{\ell}{\approx} s\ell_0 \arrow[r, hook] & s\ell
\end{tikzcd}
$$

Regular Category

$T = \mathcal{R}el(F)$

$\ell X = \ell X^{\wedge} = \ell X^{\vee}$

Satisfy all conditions in Tannaka General, Thus

$H = \mathrm{End}^{\vee}(T)$ is a commutative Hopf algebra

By definition $\mathrm{End}^{\vee}(T)$ satisfies a universal property in the category of sup-lattices

$$
\forall \quad
\begin{tikzcd}
X \arrow[d, "R"] \\
X'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TX \times TX \arrow[dr, "\langle 1 \rangle_X"] \arrow[drr, bend left, "\varphi_X"] & & \\
TX \times TX' \arrow[ur, "\square \times TR^{op}"] \arrow[dr, "TR \times \square"'] & \equiv & \mathrm{End}^{\vee}(T) \arrow[r, dashed, "\varphi"] & L \\
 & TX' \times TX' \arrow[ur, "\langle 1 \rangle_{X'}"'] \arrow[urr, bend right, "\varphi_{X'}"'] & &
\end{tikzcd}
$$

$\boxed{\Diamond\text{-cone}}$

$\langle 1 \rangle_X$, φ_X relations, $\exists!_0\, \varphi$ linear

By general Tannaka $\langle x | x' \rangle_X * \langle y | y' \rangle_Y = \langle (x\, y) | (x'\, y') \rangle_{X \times Y}$

Since $\langle (x\, x) | (x'\, x') \rangle_{X \times X} = \langle x | x' \rangle_X$, $*$ is idempotent

Thus $\mathrm{End}^{\vee}(T)$ is a localic group.

Theorem $\overset{\text{small}}{\mathcal{C}} \hookrightarrow \mathcal{E}$ generators, $\Diamond$-cone over $\mathcal{C}$ suffices.

$\Rightarrow \mathrm{End}^{\vee}(T)$ exists

The isomorphism $Aut(F) \approx End^{\vee}(T)$

Note that $T = F$ on $\mathcal{E}$, $TX = FX$ $\;T(\Gamma_f) = F(f)$.

Consider families of relations $TX \times TX \xrightarrow{\lambda_X} L$

Prop L locale, Any $\triangleright$-cone of bijections is a $\diamond$-cone.

$\Rightarrow \exists!\ \sigma$ linear $End^{\vee}(T) \xrightarrow{\sigma} Aut(F)$

Prop Elements of the form $\langle x|x'\rangle_X$ generate $\Rightarrow$ σ is a locale morphism.

Prop L locale, any $\diamond$-cone is a $\triangleright$-cone

The equation
$\lambda_X(xx') \wedge \lambda_Y(yy') = \lambda_{X\times Y}((x,y)(x',y'))$ $\Rightarrow$ λ_X are bijections

$\Rightarrow \exists!$ locale morphism $Aut(F) \xrightarrow{\rho} End^{\vee}(T)$

Prop It can be checked that σ and ρ are localic group morphisms

Theorem II

given a pointed regular category $\mathcal{E} \xrightarrow{F} \mathfrak{S}$, $T = \mathcal{R}el(F)$, there's a unique localic group isomorphism

$$\begin{tikzcd} Aut(F) \arrow[rr, "\approx"] & & End^{\vee}(T) \\ & TX \times TX \arrow[ul, "\langle 1\rangle"] \arrow[ur, "\langle 1\rangle"'] \arrow[u, phantom, "\equiv"] & \end{tikzcd}$$

$$\begin{tikzcd}
\beta^G \arrow[r] \arrow[ddr] & \mathrm{Rel}(\beta^G) \arrow[r, "\mathrm{I}", "\approx"'] & \mathrm{Com}_0(G) \arrow[r, "\mathrm{II}", "\approx"'] & \mathrm{Com}_0(H) \arrow[ddl] \\
 & \mathcal{E} \arrow[ul, "\hat{F}"] \arrow[r] \arrow[d, "F"] & \mathrm{Rel}(\mathcal{E}) \arrow[ul, "\mathrm{Rel}(\hat{F})"'] \arrow[ur, "\hat{T}"'] \arrow[d, "T"] & \\
 & \mathcal{S} \arrow[r] & \mathrm{Rel} \approx sl_0 \subset sl &
\end{tikzcd}$$

$T = \mathrm{Rel}(F)$, $G = \mathrm{Aut}(F)$, $H = \mathrm{End}^{\vee}(T)$

Theorem $\boxed{\hat{F} \text{ equivalence } \Leftrightarrow \hat{T} \text{ equivalence}}$

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$$\begin{tikzcd}
 & sl\text{-}\mathrm{Cat} \arrow[dl, bend right, "\mathrm{Map}"'] \\
 & \mathcal{D} \arrow[u, hook] \\
\mathrm{TOP}^{op} \arrow[uur, bend right, "\mathrm{Rel}"] \arrow[ur, "\approx"'] &
\end{tikzcd}$$

$\mathcal{D}$ full image of Rel

Bounded Complete Distributive Category of Relations

$$\begin{array}{ccc} \mathcal{E} & & \mathcal{A} \\ \downarrow F & \sim & \downarrow T \\ \mathcal{S} & & sl_0 \subset sl \end{array}$$

$\mathcal{E}$ connected
$\mathcal{E}$ atomic
F open
F surjective
Th A $\Updownarrow$
$\hat{F}$ equivalence

$\mathcal{A}$ connected
$\mathcal{A}$ atomic
T open
T faithful
Th B $\Updownarrow$
$\hat{T}$ equivalence

Th A is equivalent to Th B

Since Th A holds we have Th B. Tannakian Recognition

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