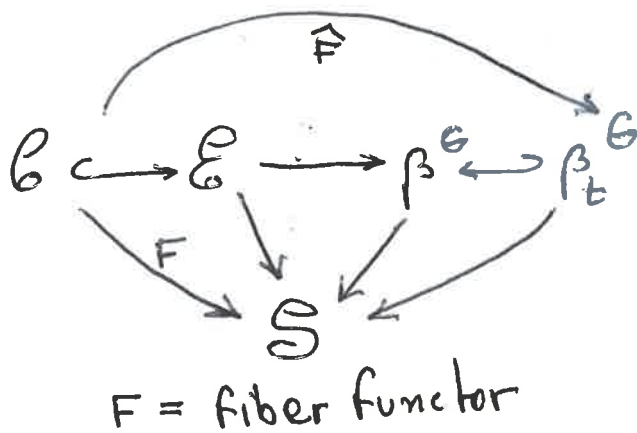


$$A = \bigwedge_{z \in FX} X$$

↳ Galois Closure

GALOIS THEORY

Theorem \hat{F} is an equivalence



\mathcal{B} site of connected objects

\mathcal{E} atomic topos (connected)

F inverse image of a point
(recall F preserves finite limits)

G automorphism group of F

GALOIS-GALOIS

F preserves all limits

$\equiv F$ is representable $F = [A_0, -]$

$G = \text{Aut}(A_0)^{op}$, G discrete

SGA 1

F finite set valued

\Rightarrow exists Galois closure

$G = \varprojlim_{(a, A)} \text{Aut}(A)^{op}$ $\text{Aut}(A)$ finite
 G profinite

SGA 4

Galois Topos

F preserves colimits

\Rightarrow exists Galois closure

G is strict progroup $\{\text{Aut}(A)\}_{(a, A)}^{op}$

$\equiv G = \varprojlim_{(a, A)} \text{Aut}(A)^{op}$ in localic groups
 G prodiscrete localic

LOCALIC GALOIS

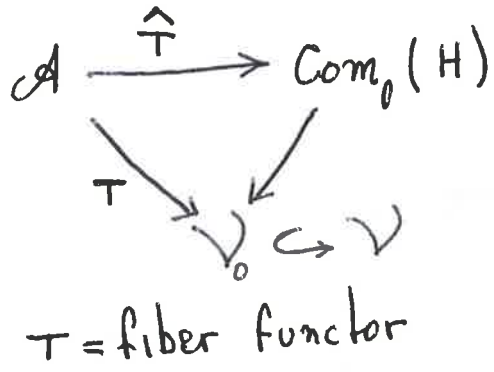
Atomic Topos

F any point

$G =$ localic group of automorphisms

G is a localic group

TANNAKA THEORY (general)



V cocomplete monoidal closed
 V_0 objects with right dual
 $H = \text{coalgebra of endomorphisms of } T$

Nat pre-dual (Loyal terminology)

$$\text{Nat}^V(L, T) = \int^X \underbrace{LX \otimes (TX)^\wedge}_{\text{hom}(LX, TX)^\wedge}$$

it follows $\text{Nat}(L, T) = \text{Nat}^V(L, T)^*$

$$H = \text{End}^V(T) = \text{Nat}^V(T, T)$$

$$H^* = V(H, 2) = \text{Nat}(T)$$

i) H is a coalgebra, and there is $TX \rightarrow \text{End}^V(T) \otimes TX$ comodule structure

ii) \mathcal{A} monoidal, T monoidal $\Rightarrow H$ bialgebra

iii) \mathcal{A} symmetric $\Rightarrow H$ is commutative algebra

iv) All objects of \mathcal{A} have duals $\Rightarrow H$ is a Hopf algebra

Alternative terminology \mathcal{A} is rigid symmetric tensor

Example $V = \text{Vec}_K$ $V_0 = V_K^{<\infty}$

Theorem \mathcal{A} abelian $\Rightarrow \hat{T}$ equivalence
 T faithful

TERMINOLOGY

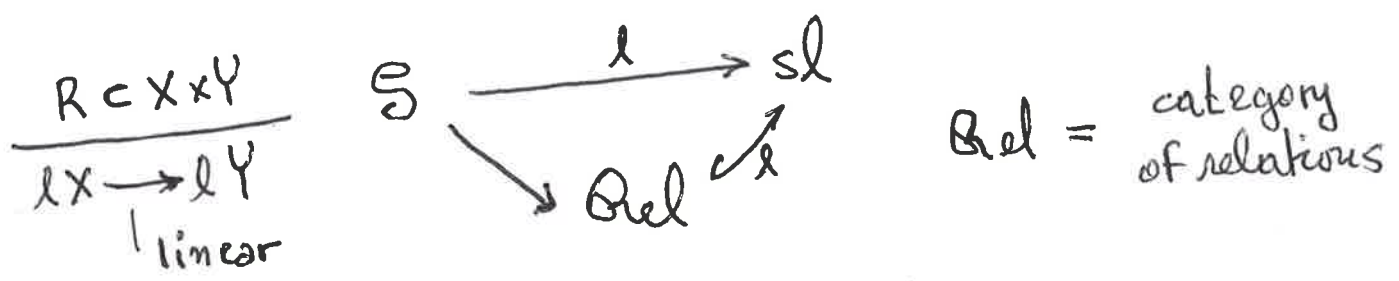
sl monoidal category of sup lattices with linear maps (sup preserving)

Locale . idempotent algebra in sl

Localic group group object in sl^{op}

Localic group \equiv Idempotent Hopf algebra

$X \in \mathcal{S}$ $\mathcal{L}X$ power set with direct image is the free sup lattice on X

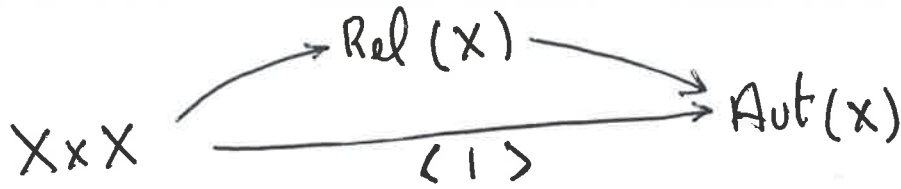


Bijections with values in a locale

- $X \times Y \xrightarrow{\lambda} \Theta$
- ed. $\bigvee_y \lambda(x, y) = 1$ each x
 - uv. $\lambda(x, y) \wedge \lambda(x, y') = 0$ $y \neq y'$
 - sv. $\bigvee_x \lambda(x, y) = 1$ each y
 - in. $\lambda(x, y) \wedge \lambda(x', y) = 0$ $x \neq x'$

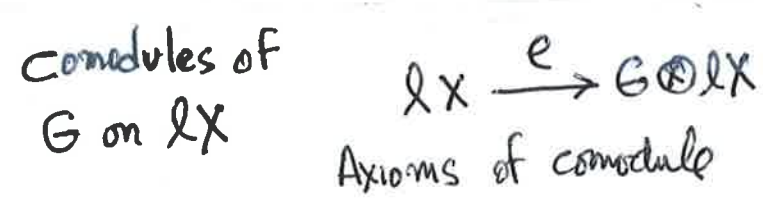
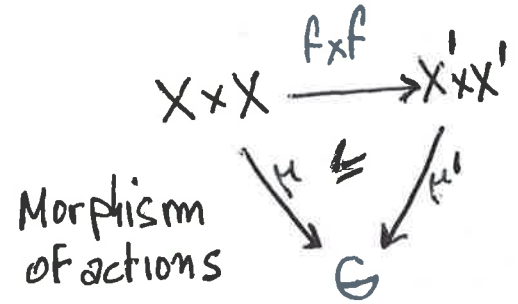
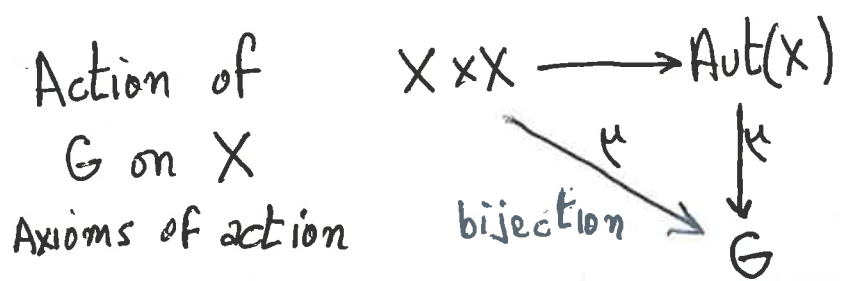
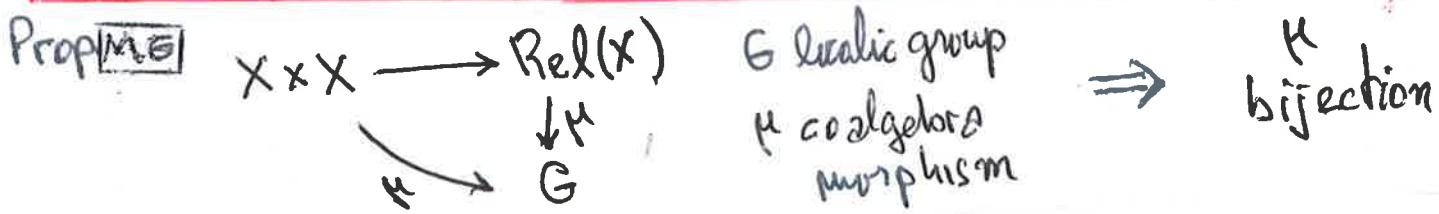
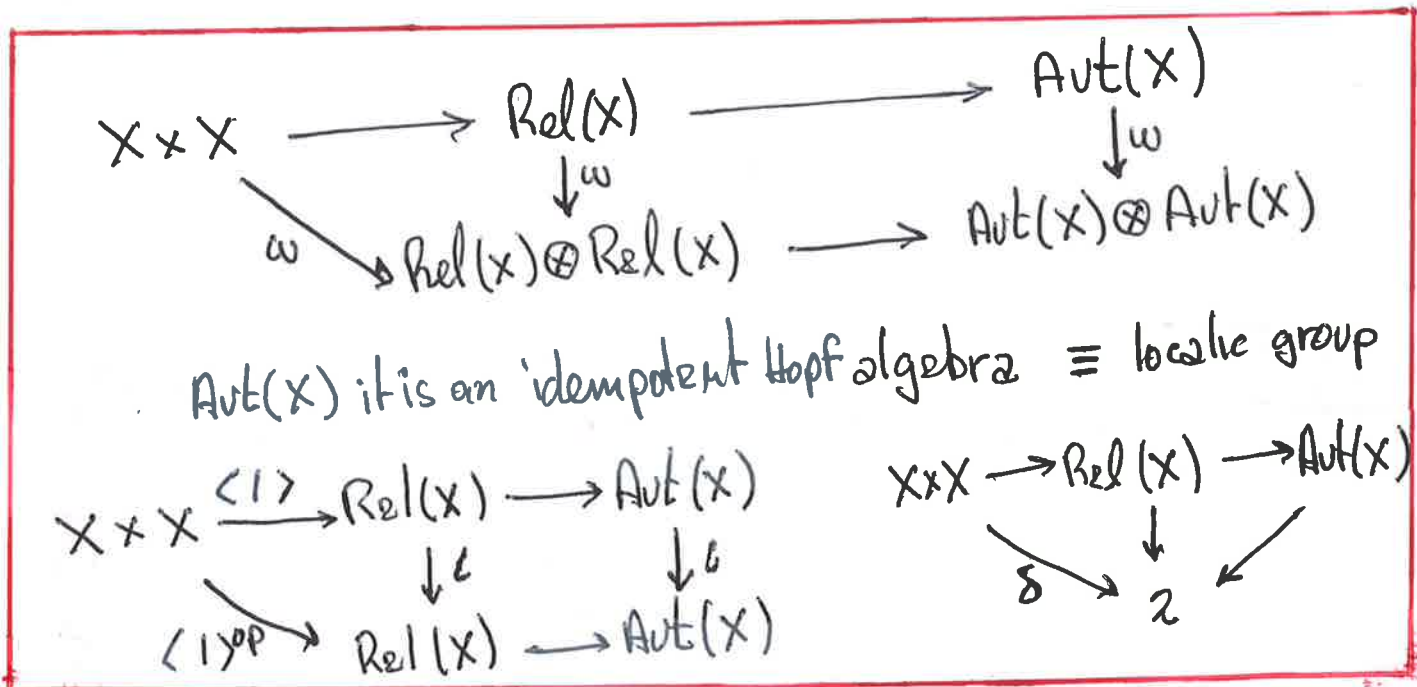
ACTIONS VERSUS COMODULES

Localic group of automorphisms $\stackrel{\text{def}}{=} \text{universal bijection}$



$\langle x, y \rangle$ is the subbase of the product topology

Coalgebra structure $\omega(x, y) = \bigvee_z (x|z) \otimes (z|y)$



Morphism of comodules as usual

G action of G on $X \equiv G$ comodule structure on $\mathcal{L}X$

Proof

$$\begin{array}{l} \text{Rel}(X) \xrightarrow{\mu} G \quad \text{locale morphism} \\ \hline X \times X \xrightarrow{\mu} G \quad \text{relation} \\ \hline \mathcal{L}X \otimes \mathcal{L}X \xrightarrow{\mu} G \quad \text{linear} \\ \hline \mathcal{L}X \xrightarrow{\rho} G \otimes \mathcal{L}X \quad \text{linear} \end{array}$$

Can check ρ G comodule $\Leftrightarrow \mu$ coalgebra morphism
 $\Uparrow \text{Prop } \boxed{MG}$
 μ G action

Also Relations between G sets $R \rightsquigarrow X \times Y$ correspond Morphisms of comodules $\mathcal{L}X \rightarrow \mathcal{L}Y$

Theorem I There is an isomorphism of sl-categories

$$\begin{array}{ccc} \text{Rel}(\beta^G) & \xrightarrow{\cong} & \text{Com}_0(G) \\ & \searrow & \swarrow \\ & \text{Rel} \xrightarrow{\mu} \text{sl}_0 & \end{array}$$

$$\frac{R \rightsquigarrow X \times Y \text{ in } \beta^G}{\mathcal{L}X \rightarrow \mathcal{L}Y \text{ comodule morphism}}$$

$$\frac{R \rightsquigarrow X \times Y}{\mathcal{L}X \rightarrow \mathcal{L}Y \text{ linear}}$$

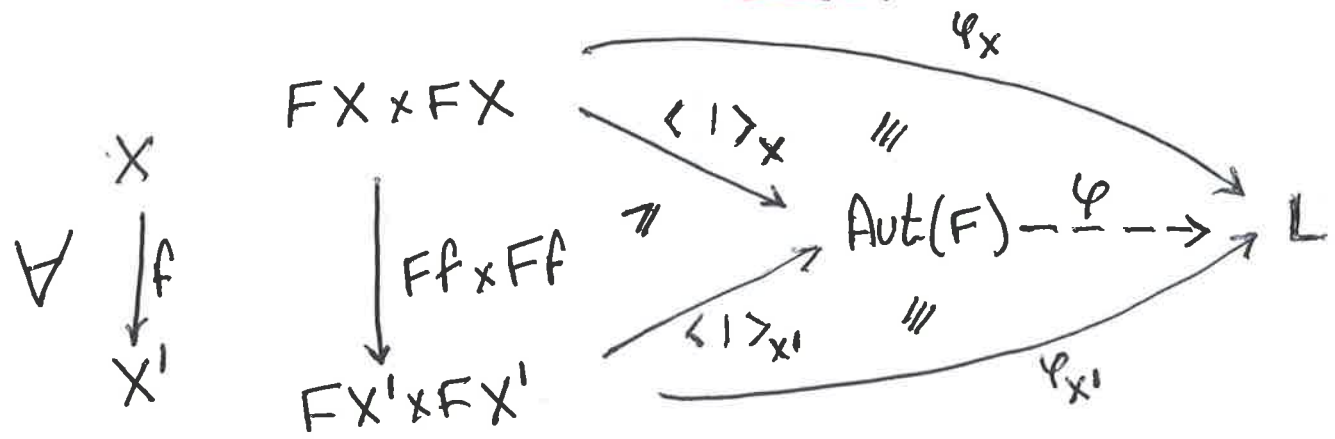
lifts

LOCALIC GALOIS THEORY

Localic Group of Automorphisms of F

is defined by a universal property in the category of Locales

$$\mathcal{E} \xrightarrow{F} \mathcal{S}$$



▷-cone

$\langle 1 \rangle_x \varphi_x$ bijections, $\exists! \varphi$ locale morphism.

$\text{Aut}(F) \rightarrow \mathbb{2}$ locale morphism \equiv Natural isomorphism of F

$$FX \times FX \xrightarrow{\omega_x} \text{Aut}(F) \otimes \text{Aut}(F), \quad FX \times FX \xrightarrow{\langle 1 \rangle_x} \text{Aut}(F), \quad FX \times FX \xrightarrow{\delta_x} \mathbb{2}$$

Defined as before are ▷-cones of bijections

Follows Aut(F) is a localic group and $\langle 1 \rangle_x$ is an action of $\text{Aut}(F)$ on FX , and $F(F)$ is a morphism of actions.

There is a lifting
$$\begin{array}{ccc} \hat{F} & \xrightarrow{\beta} & \mathbb{G} \\ \mathcal{E} & \xrightarrow{F} & \mathcal{S} \end{array} \quad \mathbb{G} = \text{Aut}(F)$$

Theorem $\mathcal{E} \hookrightarrow \mathcal{S}$ small generators, ▷-cone over \mathcal{E} suffices $\Rightarrow \text{Aut}(F)$ exists

Theorem \mathcal{E} connected atomic, \mathcal{E} connected generators $\Rightarrow \hat{F}$ equivalence

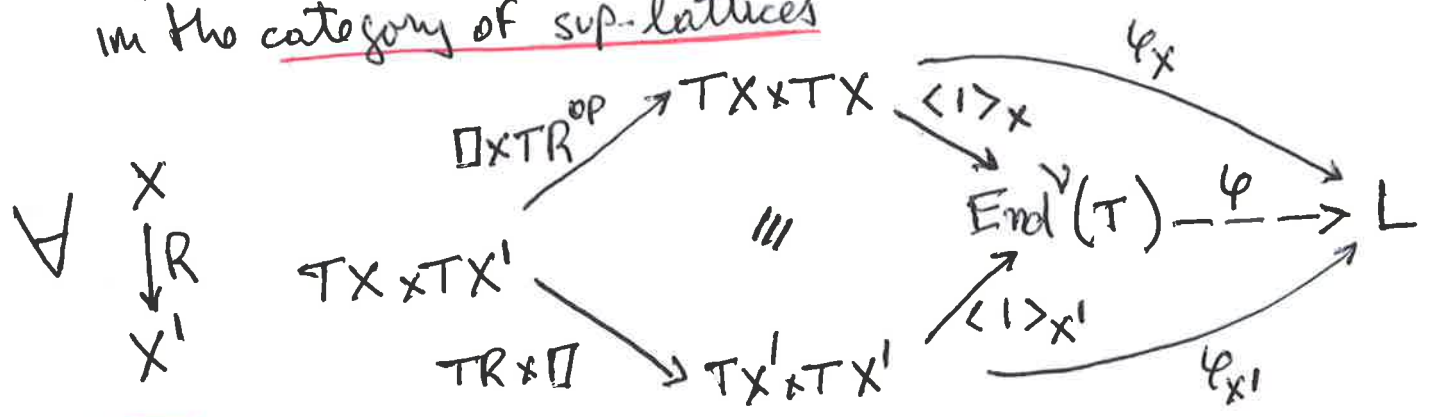
TANAKA CONTEXT FOR GALOIS

$$\begin{array}{ccc}
 \mathcal{E} & \longrightarrow & \text{Rel}(\mathcal{E}) \\
 \downarrow F & & \downarrow T \\
 \mathcal{S} & \longrightarrow & \text{Rel} \approx \mathfrak{sl}_0 \hookrightarrow \mathfrak{sl}
 \end{array}
 \quad \begin{array}{l}
 \text{Regular Category} \\
 T = \text{Rel}(F) \\
 l_X = l_{X^\wedge} = l_{X^\vee}
 \end{array}$$

Satisfy all conditions in Tannaka General, Thus

$H = \text{End}^\vee(T)$ is a commutative Hopf algebra

By definition $\text{End}^\vee(T)$ satisfies a universal property in the category of sup-lattices



◇-cone

$\langle 1 \rangle_X, \varphi_X$ relations, $\exists!$ φ linear

By general Tannaka $\langle x|x' \rangle_X * \langle y|y' \rangle_Y = \langle (x y) | (x' y') \rangle_{X \times Y}$

Since $\langle (x x) | (x' x') \rangle_{X \times X} = \langle x|x' \rangle_X$, $*$ is idempotent

Thus $\text{End}^\vee(T)$ is a localic group.

Theorem $\mathcal{B} \xrightarrow{\text{small}} \mathcal{E}$ generators, \diamond -cone over \mathcal{B} suffices.
 $\Rightarrow \text{End}^\vee(T)$ exists

8

The isomorphism $\text{Aut}(F) \cong \text{End}^V(T)$

Note that $T=F$ on \mathcal{E} , $TX=FX$ $T(\Gamma_A)=F(f)$.

Considers families of relations $TX \times TX \xrightarrow{\lambda_x} L$

Prop L locale, Any Δ -cone of bijections is a \diamond -cone.

$\Rightarrow \exists! \sigma$ linear $\text{End}^V(T) \xrightarrow{\sigma} \text{Aut}(F)$

Prop Elements of the form $\langle x|x' \rangle_x$ generate $\Rightarrow \sigma$ is a locale morphism.

Prop L locale, any \diamond -cone is a Δ -cone

The equation $\lambda_x(xz') \wedge \lambda_y(yz') = \lambda_{x \times y}(xz)(z'y')$ $\Rightarrow \lambda_x$ are bijections

$\Rightarrow \exists! \rho$ locale morphism $\text{Aut}(F) \xrightarrow{\rho} \text{End}^V(T)$

Prop It can be checked that σ and ρ are localic group morphisms

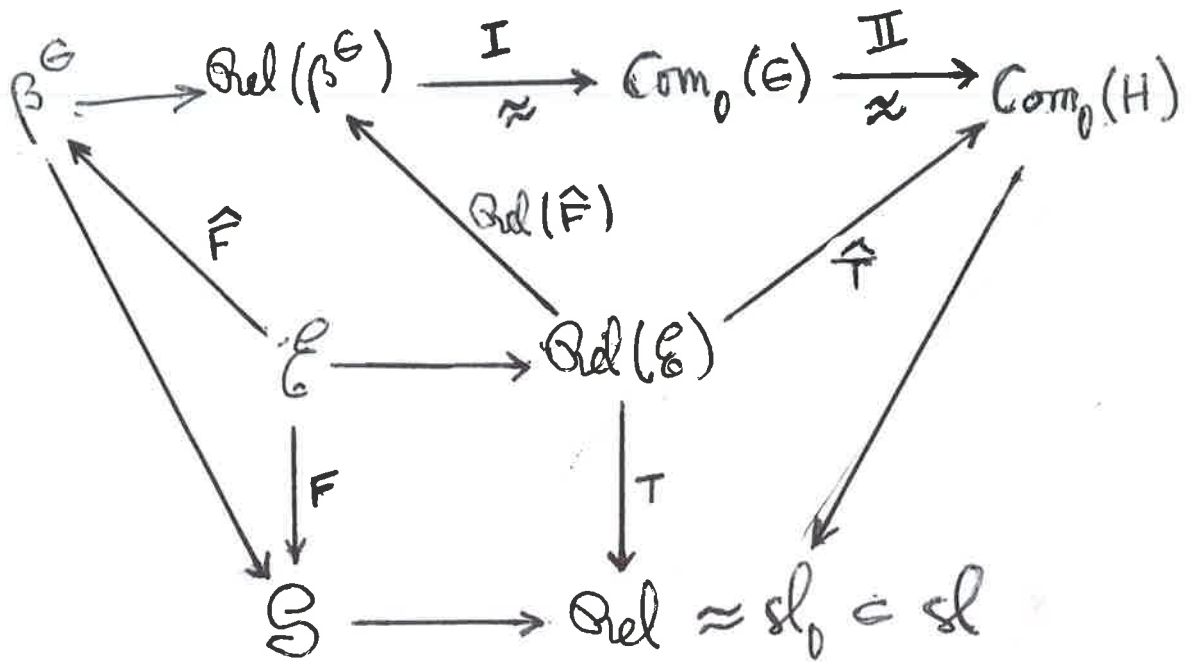
Theorem II

given a pointed regular category

$\mathcal{E} \xrightarrow{F} \mathcal{S}$, $T = \text{Obel}(F)$,

there is a unique localic group isomorphism

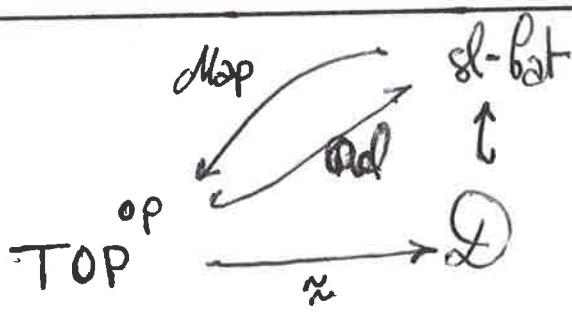
$$\begin{array}{ccc} \text{Aut}(F) & \cong & \text{End}^V(T) \\ \langle 1 \rangle \uparrow & \cong & \uparrow \langle 1 \rangle \\ & & TX \times TX \end{array}$$



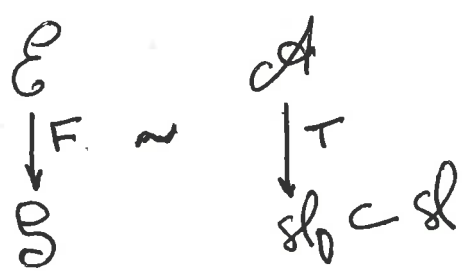
$T = \text{Rel}(F), G = \text{Aut}(F), H = \text{End}^V(T)$

Theorem \hat{F} equivalence $\Leftrightarrow \hat{T}$ equivalence

10



\mathcal{D} full image of Rel
 Bounded Complete
 Distributive Category of relations



\mathcal{E} connected
 \mathcal{E} atomic
 F open
 F surjective
 $\text{Th } A \Updownarrow$
 \hat{F} equivalence

\mathcal{A} connected
 \mathcal{A} atomic
 T open
 T faithful
 $\text{Th } B \Updownarrow$
 \hat{T} equivalence

$\text{Th } A$ is equivalent to $\text{Th } B$

Since $\text{Th } A$ holds we have $\text{Th } B$. Tannakian Recognition

Como 2018 REFERENCES (ordered by pages)

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