Grothendieck toposes as ‘bridges’ between theories

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The unifying notion of topos

"It is the topos theme which is this "bed" or "deep river" where come to be married geometry and algebra, topology and arithmetic, mathematical logic and category theory, the world of the "continuous" and that of "discontinuous" or discrete structures. It is what I have conceived of most broad to perceive with finesse, by the same language rich of geometric resonances, an "essence" which is common to situations most distant from each other coming from one region or another of the vast universe of mathematical things".

A. Grothendieck

In this course we shall present a number of ideas and techniques which allow to give substance to Grothendieck's vision by building in particular on the notion of classifying topos educed by categorical logicians.

Toposes as unifying 'bridges'

The possibility of representing a topos in a multitude of different ways can be effectively exploited for building unifying 'bridges' between theories having an equivalent, or strictly related, mathematical content.

These 'bridges' allow effective and often deep transfers of notions, ideas and results across the theories.

In spite of the number of applications in different fields obtained throughout the last years, the potential of these methods has just started to be explored.

In fact, 'bridges' have proved useful not only for connecting different theories with each other, but also for working inside a given mathematical theory and investigating it from a multiplicity of different points of view.

Plan of the course

- Background
- The 'bridge-building' technique: its key principles and the underlying vision
- Selected examples of 'bridges'
- Future perspectives

Grothendieck toposes

Definition

- A Grothendieck topos is a category (equivalent to the category) $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a site $(\mathscr{C}, J)$.
- A site is a pair $(\mathscr{C}, J)$ consisting of a small category $\mathscr{C}$ and a Grothendieck topology J on $\mathscr{C}$.

  The notion of site is a categorification of the usual notion of covering in Topology of an open set of a topological space by a family of open subsets; in fact, the definition of a sheaf on a site generalizes the classical definition of a sheaf on a topological space.

A geometric morphism of toposes $f : \mathscr{E} \to \mathscr{F}$ is a pair of adjoint functors whose left adjoint (called the inverse image functor) $f^* : \mathscr{F} \to \mathscr{E}$ preserves finite limits.

For instance, the inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ of a Grothendieck topos $\mathbf{Sh}(\mathscr{C}, J)$ in the corresponding presheaf topos $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ yields a geometric morphism between these toposes (whose inverse image is the associated sheaf functor).

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Topos-theoretic model theory

- Thanks to the rich categorical structure present on a Grothendieck topos (i.e. small limits and colimits, exponentials and a subobject classifier), one can consider models of any kind of first-order theory inside a given topos.
- The notion of model of a first-order theory in a topos is a natural generalization of the usual Tarskian definition of a (set-based) model of the theory: sorts are interpreted as objects, function symbols as arrows and relation symbols as subobjects.
- Geometric morphisms of toposes do not in general transform models of a first-order theory $\mathbb{T}$ into models of $\mathbb{T}$, unless $\mathbb{T}$ is geometric (as defined below).

Geometric theories

Definition

- A geometric formula over a signature Σ is any formula (with a finite number of free variables) built from atomic formulae over Σ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.
- A geometric theory over a signature Σ is any theory whose axioms are of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are geometric formulae over Σ and $\vec{x}$ is a context suitable for both of them.

Fact

Most of the theories naturally arising in Mathematics are geometric; and if a finitary first-order theory is not geometric, we can always associate to it a finitary geometric theory over a larger signature (the so-called Morleyization of the theory) with essentially the same models in the category **Set** *of sets.*

The syntactic category of a geometric theory

Definition (Makkai and Reyes 1977)

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. The syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the 'renaming'-equivalence classes of geometric formulae-in-context $\{\vec{x} \,.\, \phi\}$ over Σ and as arrows $\{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, \psi\}$ (where the contexts $\vec{x}$ and $\vec{y}$ are supposed to be disjoint without loss of generality) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\vec{x}, \vec{y})$ which are $\mathbb{T}$-provably functional i.e. such that the sequents
$$(\phi \vdash_{\vec{x}} (\exists y)\theta),$$
$$(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi), \text{ and}$$
$$((\theta \wedge \theta[\vec{z}/\vec{y}]) \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{y} = \vec{z}))$$
are provable in $\mathbb{T}$.
- The composite of two arrows
$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\theta]} \{\vec{y} \,.\, \psi\} \xrightarrow{[\gamma]} \{\vec{z} \,.\, \chi\}$$
is defined as the $\mathbb{T}$-provable-equivalence class of the formula $(\exists \vec{y})\theta \wedge \gamma$.
- The identity arrow on an object $\{\vec{x} \,.\, \phi\}$ is the arrow
$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\phi \wedge \vec{x'} = \vec{x}]} \{\vec{x'} \,.\, \phi[\vec{x'}/\vec{x}]\}$$

The syntactic topology

On the syntactic category of a geometric theory it is natural to put the Grothendieck topology defined as follows:

Definition

The syntactic topology $J_{\mathbb{T}}$ on the syntactic category $\mathscr{C}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ is the geometric topology on it; in particular,

a small sieve $\{[\theta_i] : \{\vec{x}_i \,.\, \phi_i\} \to \{\vec{y} \,.\, \psi\}\}$ in $\mathscr{C}_{\mathbb{T}}$ is $J_{\mathbb{T}}$-covering

if and only if

the sequent $(\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x}_i)\theta_i)$ is provable in $\mathbb{T}$.

This notion is instrumental for identifying the models of the theory $\mathbb{T}$ in any geometric category $\mathscr{C}$ (and in particular in any Grothendieck topos) as suitable functors defined on the syntactic category $\mathscr{C}_{\mathbb{T}}$ with values in $\mathscr{C}$; indeed, these are precisely the $J_{\mathbb{T}}$-continuous cartesian functors $\mathscr{C}_{\mathbb{T}} \to \mathscr{C}$.

The notion of classifying topos

Definition

Let $\mathbb{T}$ be a geometric theory over a given signature. A classifying topos of $\mathbb{T}$ is a Grothendieck topos $\mathscr{E}_{\mathbb{T}}$ such that for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

natural in $\mathscr{E}$.

It was realized in the seventies that:

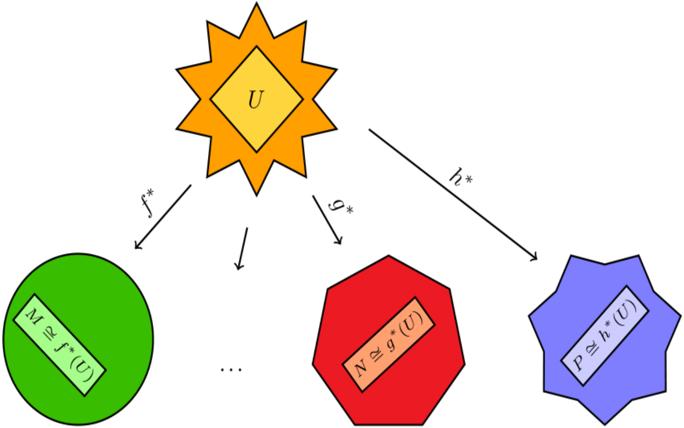
Theorem (Makkai, Reyes *et al.*)

Every geometric theory has a classifying topos. Conversely, every Grothendieck topos arises as the classifying topos of some geometric theory.

The classifying topos of a geometric theory $\mathbb{T}$ can always be constructed canonically from the theory as the topos of sheaves $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ on the syntactic site $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$.

Notice that the classical set-based models of $\mathbb{T}$ correspond to the points of its classifying topos $\mathscr{E}_{\mathbb{T}}$.

Classifying toposes



Classifying topos

Morita equivalence

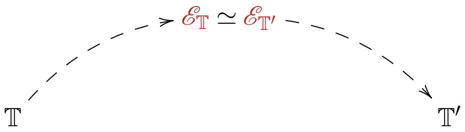
- Two mathematical theories are said to be Morita-equivalent if have the same classifying topos (up to equivalence): this means that they have equivalent categories of models in every Grothendieck topos $\mathscr{E}$, naturally in $\mathscr{E}$.
- Every Grothendieck topos is the classifying topos of *some* geometric theory (and in fact, of infinitely many theories).
- So a Grothendieck topos can be seen as a canonical representative of equivalence classes of theories modulo Morita-equivalence.
- Any two theories which are bi-interpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.
- We can expect most of the categorical equivalences between categories of set-based models of geometric theories to lift to Morita equivalences.

Toposes as *bridges*

- The notion of Morita-equivalence is ubiquitous in Mathematics; indeed, it formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods'.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted in terms of Morita-equivalences.
- On the other hand, Topos Theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.
- Any two theories which are bi-interpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.
- A mathematical theory alone gives rise to an infinite number of Morita-equivalences, through its 'internal dynamics'.

Toposes as *bridges*

- In the topos-theoretic study of theories, the latter are represented by sites (of definition of their classifying topos or of some other topos naturally attached to them), and the existence of theories which are Morita-equivalent to each other translates into the existence of different sites of definition for the same Grothendieck topos.
- Grothendieck toposes can be effectively used as 'bridges' for transferring notions, properties and results across different Morita-equivalent theories:



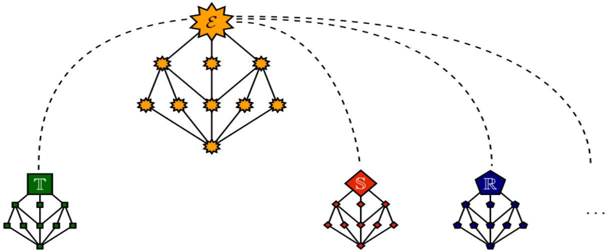
- The transfer of information takes place by expressing topos-theoretic invariants in terms of the different sites of definition (or, more generally, presentations) for the given topos.

Toposes as *bridges*

- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.
- This methodology is technically effective because the relationship between a topos and its representations is often very natural, enabling us to easily transfer invariants across different representations (and hence, between different theories).
- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories and constructions. Indeed, many important invariants of mathematical structures are actually invariants of toposes (think for instance of cohomology or homotopy groups) and topos-theoretic invariants considered on the classifying topos $\mathscr{E}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ often translate into interesting logical (i.e. syntactic or semantic) properties of $\mathbb{T}$.

Toposes as *bridges*

- The fact that topos-theoretic invariants specialize to important properties or constructions of natural mathematical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications in Mathematics as a whole: for instance

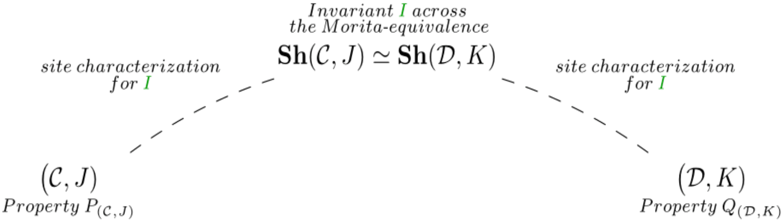


Lattices of theories

This picture represents the lattice structure on the collection of the subtoposes of a topos $\mathscr{E}$ inducing lattice structures on the collection of 'quotients' of geometric theories $\mathbb{T}$, $\mathbb{S}$, $\mathbb{R}$ classified by it.

The 'bridge-building' technique

- Decks of 'bridges': Morita-equivalences (or more generally morphisms or other kinds of relations between toposes)
- Arches of 'bridges': Site characterizations (or more generally 'unravelings' of topos-theoretic invariants in terms of concrete representations of the relevant topos)



The 'bridge' yields a logical equivalence (or an implication) between the 'concrete' properties $P_{(\mathscr{C},J)}$ and $Q_{(\mathscr{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

A mathematical morphogenesis

- The essential ambiguity given by the fact that any topos is associated in general with an infinite number of theories or different sites allows to study the relations between different theories, and hence the theories themselves, by using toposes as 'bridges' between these different presentations.
- Every topos-theoretic invariant generates a veritable mathematical morphogenesis resulting from its expression in terms of different representations of toposes, which gives rise in general to connections between properties or notions that are completely different and apparently unrelated from each other
- The mathematical exploration is therefore in a sense 'reversed' since it is guided by the Morita-equivalences and by topos-theoretic invariants, from which one proceeds to extract concrete information on the theories that one wishes to study.

Toposes as 'bridges' and the Erlangen Program

The methodology 'toposes as bridges' is a vast extension of Felix Klein's Erlangen Program (A. Joyal)

More specifically:

- Every group gives rise to a topos (namely, the category of actions of it), but the notion of topos is much more general.
- As Klein classified geometries by means of their automorphism groups, so we can study first-order geometric theories by studying the associated classifying toposes.
- As Klein established surprising connections between very different-looking geometries through the study of the algebraic properties of the associated automorphism groups, so the methodology 'toposes as bridges' allows to discover non-trivial connections between properties, concepts and results pertaining to different mathematical theories through the study of the categorical invariants of their classifying toposes.

Structural translations

The method of bridges can be interpreted linguistically as a methodology for translating concepts from one context to another.

But which kind of translation is this?

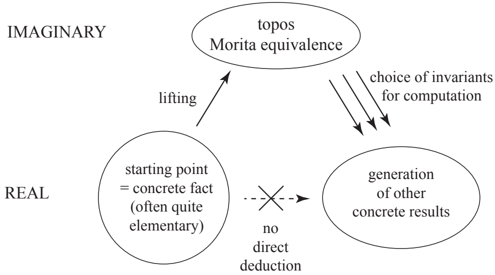
In general, we can distinguish between two essentially different approaches to translation.

- The 'dictionary-oriented' or 'bottom-up' approach, consisting in a dictionary-based renaming of the single words composing the sentences.
- The 'invariant-oriented' or 'top-down' approach, consisting in the identification of appropriate concepts that should remain invariant in the translation, and in the subsequent analysis of how these invariants can be expressed in the two languages.

The topos-theoretic translations are of the latter kind. Indeed, the invariant properties are topos-theoretic invariants defined on toposes, and the expression of these invariants in terms of the two different theories is essentially determined by the structural relationship between the topos and its two different representations.

The duality between 'real' and 'imaginary'

- The passage from a site (or a theory) to the associated topos can be regarded as a sort of 'completion' by the addition of 'imaginaries' (in the model-theoretic sense), which materializes the potential contained in the site (or theory).
- The duality between the (relatively) unstructured world of presentations of theories and the maximally structured world of toposes is of great relevance as, on the one hand, the 'simplicity' and concreteness of theories or sites makes it easy to manipulate them, while, on the other hand, computations are much easier in the 'imaginary' world of toposes thanks to their very rich internal structure and the fact that invariants live at this level.



Some examples of 'bridges'

We shall now discuss a few notable examples of 'bridges' in (and across) different areas of Mathematics:

- Topological Galois theory
- Theories of presheaf type
- 'Bridges' between quotients and topologies
- Topos-theoretic Fraïssé theorem
- Stone-type dualities
- Equivalences between lattice-ordered abelian groups and MV-algebras

The results are completely *different*... but the methodology is always the *same*!

Topological Galois theory

Recall that classical topological Galois theory provides, given a Galois extension $F \subseteq L$, a bijective correspondence between the intermediate field extensions (resp. finite field extensions) $F \subseteq K \subseteq L$ and the closed (resp. open) subgroups of the Galois group $Aut_F(L)$.

This admits the following categorical reformulation: the functor $K \to \mathrm{Hom}(K, L)$ defines an equivalence of categories

$$(\mathscr{L}_F^L)^{\mathrm{op}} \simeq \mathbf{Cont}_t(Aut_F(L)),$$

where $\mathscr{L}_F^L$ is the category of finite intermediate field extensions and $\mathbf{Cont}_t(Aut_F(L))$ is the category of continuous non-empty transitive actions of $Aut_F(L)$ on discrete sets.

A natural question thus arises: can we characterize the categories $\mathscr{C}$ whose dual is equivalent to (or fully embeddable into) the category of (non-empty transitive) actions of a topological automorphism group?

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The topos-theoretic interpretation

Key observation: the above equivalence extends to an equivalence of toposes

$$\mathbf{Sh}({\mathscr{L}_F^L}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_F(L)),$$

where J_{at} is the atomic topology on ${\mathscr{L}_F^L}^{\text{op}}$ and $\mathbf{Cont}(Aut_F(L))$ is the topos of continuous actions of $Aut_F(L)$ on discrete sets.

It is therefore natural to investigate our problem by using the methods of topos theory: more specifically, we shall look for conditions on a small category $\mathscr{C}$ and on an object u of its ind-completion for the existence of an equivalence of toposes of the form

$$\mathbf{Sh}(\mathscr{C}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u)) .$$

We will then be able to obtain, starting from such an equivalence, an answer to our question, and hence build Galois-type theories in a great variety of different mathematical contexts.

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The key notions I

- A category $\mathscr{C}$ is said to satisfy the amalgamation property (AP) if for every objects $a, b, c \in \mathscr{C}$ and morphisms $f : a \to b$, $g : a \to c$ in $\mathscr{C}$ there exists an object $d \in \mathscr{C}$ and morphisms $f' : b \to d$, $g' : c \to d$ in $\mathscr{C}$ such that $f' \circ f = g' \circ g$:

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ c & \xrightarrow[g']{} & d \end{array}$$

- A category $\mathscr{C}$ is said to satisfy the joint embedding property (JEP) if for every pair of objects $a, b \in \mathscr{C}$ there exists an object $c \in \mathscr{C}$ and morphisms $f : a \to c$, $g : b \to c$ in $\mathscr{C}$:

$$\begin{array}{ccc} & & a \\ & & \downarrow{\scriptstyle f} \\ b & \xrightarrow[g]{} & c \end{array}$$

The key notions II

- An object $u \in \text{Ind-}\mathscr{C}$ is said to be $\mathscr{C}$-universal if for every $a \in \mathscr{C}$ there exists an arrow $\chi : a \to u$ in Ind-$\mathscr{C}$:

$$a \overset{\chi}{\dashrightarrow} u$$

- An object $u \in \text{Ind-}\mathscr{C}$ is said to be $\mathscr{C}$-ultrahomogeneous if for any object $a \in \mathscr{C}$ and arrows $\chi_1 : a \to u$, $\chi_2 : a \to u$ in Ind-$\mathscr{C}$ there exists an automorphism $j : u \to u$ such that $j \circ \chi_1 = \chi_2$:

$$\begin{array}{ccc} a & \xrightarrow{\chi_1} & u \\ & \searrow^{\chi_2} & \downarrow{\scriptstyle j} \\ & & u \end{array}$$

The main theorem

Theorem

Let $\mathscr{C}$ be a small category satisfying AP and JEP, and let u be a $\mathscr{C}$-universal et $\mathscr{C}$-ultrahomogeneous object of the ind-completion Ind-$\mathscr{C}$ of $\mathscr{C}$. Then there is an equivalence of toposes

$$\mathbf{Sh}(\mathscr{C}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u)),$$

where $Aut(u)$ is endowed with the topology in which a basis of open neighbourhoods of the identity is given by the subgroups of the form $I_\chi = \{\alpha \in Aut(u) \mid \alpha \circ \chi = \chi\}$ for $\chi : c \to u$ an arrow in Ind-$\mathscr{C}$ from an object c of $\mathscr{C}$.
This equivalence is induced by the functor

$$F : \mathscr{C}^{\text{op}} \to \mathbf{Cont}(Aut(u))$$

which sends any object c of $\mathscr{C}$ to the set $\mathrm{Hom}_{\text{Ind-}\mathscr{C}}(c, u)$ (endowed with the obvious action of $Aut(u)$) and any arrow $f : c \to d$ in $\mathscr{C}$ to the $Aut(u)$-equivariant map

$$- \circ f : \mathrm{Hom}_{\text{Ind-}\mathscr{C}}(d, u) \to \mathrm{Hom}_{\text{Ind-}\mathscr{C}}(c, u)\ .$$

Restricting to sites

The following result arises from two 'bridges', respectively obtained by considering the invariant notions of atom and of arrow between atoms.

$$\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u))$$

$$\mathscr{C}^{\mathrm{op}} \qquad\qquad\qquad \mathbf{Cont}_t(Aut(u))$$

Theorem

Under the hypotheses of the last theorem, the functor F is full and faithful if and only if every arrow of $\mathscr{C}$ is a strict monomorphism, and it is an equivalence on the full subcategory $\mathbf{Cont}_t(Aut(u))$ of $\mathbf{Cont}(Aut(u))$ on the non-empty transitive actions if $\mathscr{C}$ is moreover atomically complete.

Applications

- A natural source of ultrahomogenenous and universal objects is provided by Fraïssé's construction in Model Theory and its categorical generalizations.

  For instance, if the category $\mathscr{C}$ is countable and all its arrows are monomorphisms then there always exists a $\mathscr{C}$-universal and $\mathscr{C}$-ultrahomogeneous object in Ind-$\mathscr{C}$.

- Our theorem generalizes Grothendieck's theory of Galois categories (which corresponds to the particular case when the fundamental group is profinite).

- It can be applied for generating Galois-type theories in different fields of Mathematics, which do *not* fit in the formalism of Galois categories.

Categories of 'imaginaries'

- If a category $\mathscr{C}$ satisfies the first but not the second condition of our last theorem, our topos-theoretic approach gives us a fully explicit way to complete it, by means of the addition of 'imaginaries', so that also the second condition gets satisfied.
- This is the case for instance for the categories considered above; so we get notions of 'imaginary finite set', 'imaginary finite group' etc.
- The objects of the atomic completion admit an explicit description in terms of equivalence relations in the topos $\mathbf{Sh}(\mathscr{C}^{\text{op}}, J_{at})$ on objects coming from the site $\mathscr{C}^{\text{op}}$.
- In a joint work with L. Lafforgue we give an alternative concrete description of the atomic completion.

The logical interpretation

- It is interesting to study the toposes considered above from a logical point of view, that is from the perspective of the structures that they classify.
- This analysis will reveal a deep link between Galois theory (reinterpreted and generalized as above) and Fraïssé theory in Model Theory, and lead to an approach to the problem of the independence from ℓ of ℓ-adic cohomology.
- For this, we need to recall in particular the notion of theory of presheaf type (i.e., classified by a presheaf topos).

Theories of presheaf type

Definition

A geometric theory is said to be of presheaf type if it is classified by a presheaf topos.

Theories of presheaf type are very important in that they constitute the basic 'building blocks' from which every geometric theory can be built. Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a 'quotient' of a theory of presheaf type.

These theories are the logical counterpart of small categories, in the sense that:

- For any theory of presheaf type $\mathbb{T}$, its category $\mathbb{T}$-mod(**Set**) of (set-based) models is equivalent to the ind-completion of the full subcategory f.p.$\mathbb{T}$-mod(**Set**) on the finitely presentable models.
- Any small category $\mathscr{C}$ is, up to idempotent splitting completion, equivalent to the category f.p.$\mathbb{T}$-mod(**Set**) for some theory of presheaf type $\mathbb{T}$.

Moreover, any geometric theory $\mathbb{T}$ can be expanded to a theory classified by the topos [f.p.$\mathbb{T}$-mod(**Set**), **Set**].

Theories of presheaf type

Every finitary algebraic (or, more generally, cartesian) theory is of presheaf type, but this class contains many other interesting mathematical theories including

- the theory of linear orders (classified by the simplicial topos)
- the theory of algebraic extensions of a given field
- the theory of flat modules over a ring
- the theory of lattice-ordered abelian groups with strong unit
- the 'cyclic theories' (classified by the cyclic topos, the epicyclic topos and the arithmetic topos)
- the theory of perfect MV-algebras (or more generally of local MV-algebras in a proper variety of MV-algebras)
- the geometric theory of finite sets

Any theory of presheaf type $\mathbb{T}$ gives rise to two different representations of its classifying topos, which can be used to build 'bridges' connecting its syntax and semantics:

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

$$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$$

$$(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

Irreducible formulae and finitely presentable models

Definition

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then a geometric formula $\phi(\vec{x})$ over Σ is said to be $\mathbb{T}$-irreducible if, regarded as an object of the syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$, it does not admit any non-trivial $J_{\mathbb{T}}$-covering sieves.

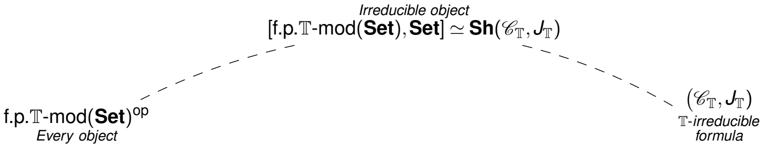
Theorem

Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ. Then

(i) *Any finitely presentable $\mathbb{T}$-model in* **Set** *is presented by a $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ;*

(ii) *Conversely, any $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ presents a $\mathbb{T}$-model.*

In fact, the category f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ *is equivalent to the full subcategory $\mathscr{C}_{\mathbb{T}}^{\text{irr}}$ of $\mathscr{C}_{\mathbb{T}}$ on the $\mathbb{T}$-irreducible formulae.*



A definability theorem

Theorem

Let $\mathbb{T}$ be a theory of preshef type and suppose that we are given, for every finitely presentable **Set***-model $\mathscr{M}$ of $\mathbb{T}$, a subset $R_{\mathscr{M}}$ of $\mathscr{M}^n$ in such a way that every $\mathbb{T}$-model homomorphism $h : \mathscr{M} \to \mathscr{N}$ maps $R_{\mathscr{M}}$ into $R_{\mathscr{N}}$. Then there exists a geometric formula-in-context $\phi(x_1, \dots, x_n)$ such that $R_{\mathscr{M}} = [[\vec{x} \,.\, \phi]]_{\mathscr{M}}$ for each finitely presentable $\mathbb{T}$-model $\mathscr{M}$.*

Subobject of $UA_1 \times \cdots \times UA_n$

$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$

Functorial assignment

$M \to R_M \subseteq MA_1 \times \cdots \times MA_n$

$(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

Geometric formula

$\phi(x_1^{A_1}, \dots, x_n^{A_n})$

Characterization theorems

Theorem

A geometric theory $\mathbb{T}$ over a signature Σ is of presheaf type if and only if every geometric formula $\phi(\vec{x})$ over Σ, when regarded as an object of $\mathscr{C}_{\mathbb{T}}$, is $J_{\mathbb{T}}$-covered by $\mathbb{T}$-irreducible formulae over Σ.

Theorem

A geometric theory $\mathbb{T}$ over a signature Σ is of presheaf type if and only if the following conditions are satisfied:

(i) *Every finitely presentable model is presented by a geometric formula over Σ.*

(ii) *Every property of finite tuples of elements of a finitely presentable $\mathbb{T}$-model which is preserved by $\mathbb{T}$-model homomorphisms is definable (in finitely presentable $\mathbb{T}$-models) by a geometric formula over Σ.*

(iii) *The finitely presentable $\mathbb{T}$-models are jointly conservative for $\mathbb{T}$.*

I have also established a characterization theorem providing necessary and sufficient semantic conditions for a theory to be of presheaf type.

'Bridges' between quotients and topologies

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the syntactic-equivalence classes of quotients (i.e. geometric theory extensions over the same signature) of $\mathbb{T}$ and the subtoposes of the classifying topos $\mathbf{Set}[\mathbb{T}]$ of $\mathbb{T}$.

This duality allows one in particular to establish 'bridges' of the following form:

Subtopos of

$\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Set}[\mathbb{T}]$

Grothendieck topology on $\mathscr{C}$ containing J — *Quotient of $\mathbb{T}$*

That is, if the classifying topos of a geometric theory $\mathbb{T}$ can be represented as the category $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a (small) site $(\mathscr{C}, J)$ then we have a natural, order-preserving bijection

$$\text{quotients of } \mathbb{T}$$
$$|||$$
$$\text{Grothendieck topologies on } \mathscr{C} \text{ which contain } J$$

Two notable cases

This result can be applied in particular in the following two cases:

(1) $(\mathscr{C}, J)$ is the syntactic site $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$

(2)
- $\mathbb{T}$ is a theory of presheaf type,
- $\mathscr{C}$ is the opposite of its category f.p.$\mathbb{T}$-mod(**Set**) of finitely presentable models, and
- J is the trivial topology on it.

In the first case, we obtain an order-preserving bijective correspondence between the quotients of $\mathbb{T}$ and the Grothendieck topologies on $\mathscr{C}_{\mathbb{T}}$ which contain $J_{\mathbb{T}}$.

In the second case, we obtain an order-preserving bijective correspondence between the quotients of $\mathbb{T}$ and the Grothendieck topologies on f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$.

In both cases, these correspondences can be naturally interpreted as proof-theoretic equivalences between the classical proof system of geometric logic over $\mathbb{T}$ and new proof systems for sieves whose inference rules correspond to the axioms of Grothendieck topologies.

Quotients of a theory of presheaf type I

The Grothendieck topology J on f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ associated with a quotient $\mathbb{T}'$ of a theory of presheaf type $\mathbb{T}$ can be explicitly described as follows.

- By using the fact that every geometric formula over Σ can be $J_{\mathbb{T}}$-covered in $\mathscr{C}_{\mathbb{T}}$ by $\mathbb{T}$-irreducible formulae, one can show that every geometric sequent over Σ is provably equivalent in $\mathbb{T}$ to a collection of sequents σ of the form $(\phi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{y_i})\theta_i)$ where, for each $i \in I$, $[\theta_i] : \{\vec{y_i} \,.\, \psi_i\} \to \{\vec{x} \,.\, \phi\}$ is an arrow in $\mathscr{C}_{\mathbb{T}}$ and $\phi(\vec{x})$, $\psi(\vec{y_i})$ are geometric formulae over Σ presenting respectively $\mathbb{T}$-models $M_{\{\vec{x} \,.\, \phi\}}$ and $M_{\{\vec{y_i} \,.\, \psi_i\}}$.
- To such a sequent σ, we can associate the cosieve S_σ on $M_{\{\vec{x} \,.\, \phi\}}$ in f.p.$\mathbb{T}$-mod(**Set**) generated by the arrows s_i defined as follows. For each $i \in I$, $[[\theta_i]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}}$ is the graph of a morphism $[[\vec{y_i} \,.\, \psi_i]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}} \to [[\vec{x} \,.\, \phi]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}}$; then the image of the generators of $M_{\{\vec{y_i} \,.\, \psi_i\}}$ via this morphism is an element of $[[\vec{x} \,.\, \phi]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}}$ and this in turn determines, by definition of $M_{\{\vec{x} \,.\, \phi\}}$, a unique arrow $s_i : M_{\{\vec{x} \,.\, \phi\}} \to M_{\{\vec{y_i} \,.\, \psi_i\}}$ in $\mathbb{T}$-mod(**Set**).
- Conversely, by the equivalence f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}} \simeq \mathscr{C}_{\mathbb{T}}^{\text{irr}}$, every sieve in f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ is of the form S_σ for such a sequent σ.

Quotients of a theory of presheaf type II

The Grothendieck topology J on f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ associated with a quotient $\mathbb{T}'$ of $\mathbb{T}$ is generated by the sieves S_σ, where σ varies among the sequents of the required form which are equivalent to the axioms of $\mathbb{T}'$.

The equivalence

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

of classifying toposes for $\mathbb{T}$ restricts to an equivalence

$$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$$

of classifying toposes for $\mathbb{T}'$.
In particular, for any σ of the above form, σ is provable in $\mathbb{T}'$ if and only if S_σ belongs to J.

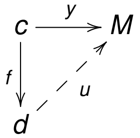
These equivalences are useful in that they allow us to study (the proof theory of) geometric theories through the associated Grothendieck topologies: the condition of provability of a sequent in a geometric theory gets transformed in the requirement for a sieve (or a family of sieves) to belong to a certain Grothendieck topology, something which is often much easier to investigate.

Topos-theoretic Fraïssé theorem

The following result, which generalizes Fraïssé's theorem in classical model theory, arises from a triple 'bridge'.

Definition

A set-based model M of a geometric theory $\mathbb{T}$ is said to be homogeneous if for any arrow $y : c \to M$ in $\mathbb{T}$-mod(**Set**) and any arrow f in f.p.$\mathbb{T}$-mod(**Set**) there exists an arrow u in $\mathbb{T}$-mod(**Set**) such that $u \circ f = y$:



Theorem

Let $\mathbb{T}$ be a theory of presheaf type such that the category f.p.$\mathbb{T}$-mod(**Set**) *is non-empty and has AP and JEP. Then the theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models is complete and atomic.*

Topos-theoretic Fraïssé theorem

Atomic topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
Atomic site i.e.
AP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Atomicity of $\mathbb{T}'$

Two-valued topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
JEP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Completeness of $\mathbb{T}'$

Point of

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
homogeneous $\mathbb{T}$-model in **Set**

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
$\mathbb{T}'$-model in **Set**

Stone-type dualities through topos-theoretic 'bridges'

All the classical Stone-type dualities/equivalences between special kinds of preorders and locales or topological spaces can be obtained by functorializing 'bridges' of the form

$$\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}}) \simeq \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}})$$

$$\mathscr{C} \qquad\qquad\qquad\qquad \mathscr{D}$$

where $\mathscr{D}$ is a $J_{\mathscr{C}}$-dense subcategory of a preorder category $\mathscr{C}$.

For instance, take $\mathscr{D}$ equal to a Boolean algebra and $\mathscr{C}$ equal to the lattice of open sets of its Stone space for Stone duality, $\mathscr{C}$ equal to an atomic complete Boolean algebra and $\mathscr{D}$ equal to the collection of its atoms for Lindenbaum-Tarski duality.

This method also allows to generate many new dualities for other kinds of pre-ordered structures (for instance, a localic duality for meet-semilattices, a duality for k-frames, a duality for disjunctively distributive lattices, a duality for preframes generated by their directedly irreducible elements etc. It also naturally generalizes to the setting of arbitrary categories.

'Bridges' between ℓ-groups and MV-algebras

With my former Ph.D. student A.C. Russo, we have :

- Lifted to Morita equivalences two well-known categorical equivalences between classes of MV-algebras and classes of lattice-ordered abelian groups, namely:
  - Mundici's equivalence:

    category of MV-algebras $\simeq$ category of ℓ-groups with strong unit
  - Di Nola-Lettieri's equivalence:

    category of perfect MV-algebras $\simeq$ category of ℓ-groups
- Applied the 'bridge' technique to these Morita equivalences;
- Built (through the study of certain classifying toposes) a new class of (Morita-)equivalences including in particular the one lifting Di Nola-Lettieri's equivalence.

Results in connection with Mundici's equivalence

- The theory of ℓ-groups with strong unit is of presheaf type (i.e. classified by a presheaf topos) and in fact Morita-equivalent to an algebraic theory (namely that of MV-algebras)
- Bijective correspondence between the quotients (i.e. geometric theory extensions over the same signature) of the theory of MV-algebras and those of the theory of ℓ-u groups (in spite of the fact that these theories are not bi-interpretable)
- Logical characterization of the finitely presentable ℓ-u groups
- Form of compactness and completeness for the geometric theory of ℓ-u groups (in spite of the infinitary nature of this theory)
- Sheaf-theoretic version of Mundici's equivalence

Results in connection with Di Nola-Lettieri's equivalence

- The theory of perfect MV-algebras is of presheaf type and in fact Morita-equivalent to an algebraic theory (namely that of ℓ-groups)
- Three levels of partial bi-interpretability for
  - irreducible formulas
  - geometric sentences
  - imaginaries
- the finitely presentable models of the theory of perfect MV-algebras are finitely presentable also with respect to the variety generated by Chang's algebra
- Representation result: every finitely presentable MV-algebra in the variety generated by Chang's MV-algebra is a finite direct product of finitely presentable perfect MV-algebras
- a Morita-equivalence (actually, bi-interpretability) between the theory of lattice-ordered abelian groups and that of cancellative lattice-ordered abelian monoids with bottom element.

Results for local MV-algebras in varieties

All the theories considered above are of presheaf type. We have shown that:

- The theory of local MV-algebras is NOT of presheaf type
- The theory of local MV-algebras in an arbitrary proper subvariety of MV-algebras IS of presheaf type
- The theory of local MV-algebras in an arbitrary proper subvariety of MV-algebras is Morita-equivalent to a theory extending that of lattice-ordered abelian groups
- the finitely presentable models of the theory of local MV-algebras in an arbitrary proper subvariety are finitely presentable also with respect to the variety
- The theory of simple MV-algebras is NOT of presheaf type, but the geometric theory of finite MV-chains IS
- Representation result: every finitely presentable MV-algebra in an abitrary proper subvariety of MV-algebras is a finite direct product of finitely presentable local MV-algebras in the variety

Future perspectives

The evidence provided by the results obtained so far shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories and for generating new equivalences, dualities and symmetries across different fields of Mathematics.

In fact, toposes have an authentic creative power in Mathematics, in the sense that their study naturally leads to the discovery of a great number of notions and 'concrete' results in different mathematical fields, which are pertinent but often unsuspected.

We are still at the beginning of explorations of methods for exploiting the unifying potential of the notion of Grothendieck topos in Mathematics; much remains to do both at the theoretical level and at the applied level, so that toposes become fundamental tools in the study of mathematical theories and their relations.

Future perspectives

Natural research directions in this respect are:

- investigation of important dualities or correspondences in Mathematics from a topos-theoretic perspective (in particular, the theory of motives, class field theory and the Langlands programme)
- systematic study of invariants of toposes in terms of their presentations, and introduction of new invariants which capture important aspects of concrete mathematical problems
- interpretation and generalization of important parts of classical and modern model theory in terms of toposes and development of a functorial model theory
- introduction of new methodologies for generating Morita-equivalences
- development of general techniques for building spectra by using classifying toposes
- generalization of the 'bridge' technique to the setting of higher categories and toposes through the introduction of higher geometric logic
- development of a relative theory of classifying toposes
- automation of the methodology 'toposes as bridges' to generate new and non-trivial mathematical results in a mechanical way

For further reading

O. Caramello
Grothendieck toposes as unifying 'bridges' in Mathematics,
Mémoire d'habilitation à diriger des recherches,
Université de Paris 7 (2016),
available from my website www.oliviacaramello.com.

O. Caramello
Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges',
Oxford University Press (2017).