"Think Topos"

in arithmetics

A. Connes

(Jt work with C. Consani)
We **Think Different** about the integers, which we view as a semiringed topos instead of an ordered ring!.

\[ \mathbb{Z}_{\text{max}} = (\mathbb{Z} \cup \{-\infty\}, \max, +) \] on which \( \mathbb{N}^\times \) acts by \( n \mapsto \text{Fr}_n \) is a semiring in the topos \( \mathbb{N}^\times \) of sets with an action of \( \mathbb{N}^\times \).

**The Arithmetic Site \( (\mathbb{N}^\times, \mathbb{Z}_{\text{max}}) \) is the topos \( \mathbb{N}^\times \) endowed with the structure sheaf** : \( \mathcal{O} := \mathbb{Z}_{\text{max}} \) semiring in the topos.
Why semirings?

A category $\mathcal{C}$ is *semiadditive* if it has finite products and coproducts, the morphism $0 \to 1$ is an isomorphism (thus $\mathcal{C}$ has a 0), and the morphisms

$$\gamma_{M,N} : M \lor N \to M \times N$$

are isomorphisms.

Then $\text{End}(M)$ is naturally a semiring for any object $M$.

Finite semifields, characteristic 1

$K = \text{finite semifield} :$ then $K$ is a field or $K = \mathbb{B} :$

$$\mathbb{B} := \{0, 1\}, \quad 1 + 1 = 1$$
The semifield $\mathbb{Z}_{\text{max}}$

**Lemma**: Let $F$ be a semifield of characteristic 1, then for $n \in \mathbb{N}^\times$ the map $\text{Fr}_n \in \text{End}(F)$, $\text{Fr}_n(x) := x^n \; \forall x \in F$ defines an **injective endomorphism** of $F$.

$\mathbb{Z}_{\text{max}} := (\mathbb{Z} \cup \{-\infty\}, \max, +)$, unique semifield with multiplicative group infinite cyclic.

*multiplicative notation*: Addition $\lor$, $u^n \lor u^m = u^k$, with $k = \sup(n, m)$. *Multiplication*: $u^n u^m = u^{n+m}$.

Map $\mathbb{N}^\times \to \text{End}(\mathbb{Z}_{\text{max}})$, $n \mapsto \text{Fr}_n$ is isomorphism of semi-groups. (extend to 0)
**Theorem**

(Connes-Consani)

Let $X$ be a smooth, projective variety of dimension $d$ over an algebraic number field $K$ and let $\nu|_\infty$ be an archimedean place of $K$. Then,

\[
\prod_{0 \leq w \leq 2d} L_\nu(H^w(X), s) (-1)^w = \frac{\text{det}_\infty(\frac{1}{2\pi}(s - \Theta))|_{HC^\text{ar,odd}(X_\nu)}}{\text{det}_\infty(\frac{1}{2\pi}(s - \Theta))|_{HC^\text{ar,even}(X_\nu)}}
\]

The left-hand side is the product of Serre’s archimedean local factors of the complex $L$-function of $X$. On the right-hand side, $\text{det}_\infty$ denotes the regularized determinant and one sets $HC^\text{ar}(X_\nu) = \bigoplus_{n=2k \geq 0} HC_n^\text{ar}(X_\nu)$, $HC^\text{ar,odd}(X_\nu) = \bigoplus_{n=2k+1 \geq 1} HC_n^\text{ar}(X_\nu)$. 

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The endomorphism $\Theta$ has two constituents: the natural grading in cyclic homology and the action of the multiplicative semigroup $\mathbb{N}^\times$ on cyclic homology of commutative algebras given by the $\lambda$-operations $\Lambda(k)$, $k \in \mathbb{N}^\times$. More precisely, the action $u^\Theta$ of the multiplicative group $\mathbb{R}_+^\times$ generated by $\Theta$ on cyclic homology, is uniquely determined by its restriction to the dense subgroup $\mathbb{Q}_+^\times \subset \mathbb{R}_+^\times$ where it is given by the formula

$$k^\Theta|_{HC_n(X_\nu)} = \Lambda(k) k^{-n}, \quad \forall n \geq 0, \ k \in \mathbb{N}^\times \subset \mathbb{R}_+^\times.$$
The $\lambda$-operations

Let $A$ be a commutative algebra and

$$(C_k(A) = A^{\otimes (k+1)}, b, B)$$

its mixed complex of chains. The $\lambda$-operations define (degree zero) endomorphisms $\Lambda_\ast$ of $C_\ast(A)$ commuting with the grading and satisfying the following properties

- $\Lambda_{nm} = \Lambda_n \Lambda_m$, $\forall n, m \in \mathbb{N}^\times$
- $b \Lambda_m = \Lambda_m b$, $\forall m \in \mathbb{N}^\times$
- $\Lambda_mB = mB\Lambda_m$, $\forall m \in \mathbb{N}^\times$.

$$HC_n(A) = \bigoplus_{j \geq 0} HC_n^{(j)}(A)$$
Cyclic homology of projective varieties

\[ HC_n(X_{\mathbb{C}}) = \bigoplus_{j \geq 0} HC_n^{(j)}(X_{\mathbb{C}}). \]

We recall the following key result of Weibel

Let \( X_{\mathbb{C}} \) be a smooth, projective algebraic variety over \( \mathbb{C} \). Then one has canonical isomorphisms \( \forall j \geq 0, \forall n \geq 0 \)

\[ HC_n^{(j)}(X_{\mathbb{C}}) \cong \tilde{H}^{2j+1-n}_D(X_{\mathbb{C}}, \mathbb{R}(j+1)) \cong H^{2j-n}_B(X(\mathbb{C}), \mathbb{C})/F^{j+1} \]
Archimedean cyclic homology

For a smooth, complex projective variety $X_{\mathbb{C}}$ and for any pair of integers $(n, j) \in E_d$, there is a short exact sequence

$$0 \rightarrow HP_{n+2}^{\text{real},(j+1)}(X_{\mathbb{C}}) \xrightarrow{S \circ \tau} HC_n^{(j)}(X_{\mathbb{C}}) \rightarrow HC_n^{\text{ar},(j)}(X_{\mathbb{C}}) \rightarrow 0.$$ 

The periodic cyclic homology is unaltered by the morphism of schemes

$$\pi_X : \text{Spec}(C^\infty(X_{\text{sm}}, \mathbb{C})) \rightarrow X_{\mathbb{C}}$$
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Linear algebra over $\mathbb{F} = \mathbb{Z}_{\text{max}}$

Module $\mathbb{F}^{(n)}$ over $\mathbb{F}$ obtained from the one dimensional free module $\mathbb{F}$ by restriction of scalars using the endomorphism $\text{Fr}_n \in \text{End}(\mathbb{F})$.

Projective space $\mathbb{P}(\mathbb{F}^{(n)})$ is finite

Submodules $E$ are in bijection with subsets of $\mathbb{P}(\mathbb{F}^{(n)})$

$E$ is isomorphic to $\mathbb{F}^{(k)}$ where $k$ is the rank of $E$
Projective Geometry

In projective geometry the maps between projective spaces \( \mathbb{P}(E_j) = (E_j \setminus \{0\})/K_j^\times \) \((j = 1, 2)\) over fields \(K_j\) are induced by *semilinear* maps \(f : E_1 \to E_2\) of vector spaces, with \(f^{-1}(\{0\}) = \{0\}\).

A map \(f : E_1 \to E_2\) is semilinear if it is additive and if there exists an injective homomorphism of fields \(\sigma : K_1 \to K_2\) such that \(f(\lambda x) = \sigma(\lambda)f(x)\) \(\forall \lambda \in K_1\) and \(\forall x \in E_1\). This notion extends verbatim to the context of semifields.
The epicyclic category $\tilde{\Lambda}$ is canonically isomorphic to the category $\mathcal{P}_F$ whose objects are the modules $F^{(n)}$ over $F$ and the morphisms the projective classes of non-zero semilinear maps. (Here $F = \mathbb{Z}_{\text{max}}$)

The cyclic category $\Lambda \subset \tilde{\Lambda}$ is isomorphic to the subcategory $\mathcal{P}^1_F \subset \mathcal{P}_F$ with the same objects of $\mathcal{P}_F$ and whose morphisms are induced by linear maps.
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<td>Epicyclic $\tilde{\Lambda} \sim \mathcal{P}_\mathbb{F}$</td>
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Self-duality of $\Lambda = \text{transposition}$

The following pairing is well defined with values in $\mathbb{F} = \mathbb{Z}_{\text{max}}$:

$$< x, y >_{\mathbb{F}} := \inf \{ v \in \mathbb{F} \mid x \leq vy \}, \quad \forall x, y \in E, y \neq 0.$$  

The contravariant functor $\Lambda \longrightarrow \Lambda$, $f \mapsto f^t$ is the inverse of transposition $f \mapsto f^*$:

$$\text{Hom}_\mathbb{F}(E, F) \ni f \mapsto f^* \in \text{Hom}_\mathbb{F}(F^*, E^*)$$

$$< f(x), y >_{\mathbb{F}} = < x, f^*(y) >_{\mathbb{F}}, \quad \forall x \in E, y \in F^*.$$
Cyclic descent number

(i) The functor $\mathbb{P} : \mathcal{P}_F \rightarrow \text{Fin}$ is full.
(ii) Let $\sigma \in \text{Hom}_{\text{Fin}}([n], [m])$. Then $\text{cdesc}(\sigma)$ is the smallest integer $k$ such that there exists $f \in \text{Hom}_{\mathcal{P}_F}(F(n), F(m))$, $\text{Mod}(f) = k$ with $\mathbb{P}(f) = \sigma$.
(iii) Let $\sigma \in \text{Hom}_{\text{Fin}}([n], [m])$ with $\text{cdesc}(\sigma) = k$, then there exists a unique $f \in \text{Hom}_{\mathcal{P}_F}(F(n), F(m))$, $\text{Mod}(f) = k$ such that $\mathbb{P}(f) = \sigma$. 
The epicyclic site

When working with the $\lambda$-operations the relevant topos is the *epicyclic topos* $(\tilde{\Lambda}^{\text{op}})^\wedge$ which is the topos of *covariant* functors from the epicyclic category to the category $\mathbf{Sets}$ of sets. By construction this is the dual of the opposite of the epicyclic category $\tilde{\Lambda}$. We recall that to a commutative ring corresponds canonically an epicyclic module.

**Definition**: An *epicyclic module* $E$ is a *covariant* functor from the epicyclic category $\tilde{\Lambda}$ to the category of abelian groups.

Thus once translated in the language of topoi, an epicyclic module is nothing but a sheaf of abelian groups.
The following general result shows why the epicyclic topos encodes the $\lambda$-operations

**Theorem**: For any epicyclic module $E$, the $\lambda$-operations define an action of $\mathbb{N}^\times$ on the cyclic homology groups $HC_\ast(E)$.

When the epicyclic module $E$ factors through the category $\mathcal{F}_{\text{fin}}$ of finite sets one deduces a decomposition with integral weights

$$HC_n(E) = \bigoplus_{j \geq 0} HC_n^{(j)}(E).$$
In general the action of $\mathbb{N}^\times$ on the cyclic homology groups may have non-integral weights as well. Indeed, the tensor product of an epicyclic module with a representation of $\mathbb{N}^\times$ is still an epicyclic module and one can produce in this way examples of epicyclic modules where the weights are complex numbers. In particular, one can twist an epicyclic module by a Dirichlet character modulo $m$, (i.e. a character of the multiplicative group $(\mathbb{Z}/m\mathbb{Z})^\times$) since any such character can be viewed as a multiplicative map $\mathbb{N}^\times \rightarrow \mathbb{C}$. 
Points of the epicyclic topos

The next result shows the relevance of geometry in characteristic 1 for the epicyclic topos.

**Theorem** : The category of points of the epicyclic topos \((\tilde{\Lambda}^{\text{op}})^{\wedge}\) is equivalent to the category \(\mathcal{P}\) whose objects are pairs \((K, E)\) where \(K\) is an algebraic extension of \(F = \mathbb{Z}_{\text{max}}\) and \(E\) is an archimedean semimodule over \(K\). The morphisms are projective classes of semilinear maps.
Relation between $\hat{\Lambda}^{\text{op}}$ and $\mathbb{N}^X$

The functor $\text{Mod} : \hat{\Lambda}^{\text{op}} \to \mathbb{N}^X$ gives a geometric morphism of toposes and one has a natural section $\iota$ using $\text{Hom}_{\hat{\Lambda}}(0, 0) \cong \mathbb{N}^X$, $\text{Mod} \circ \iota = \text{Id}$. 
Conjectural role of $HC_\ast$
and of $\lambda$-operations

The relations between the topos $\tilde{\Lambda}^{\text{op}}$ and $\hat{N}^X$ and with the BC system suggest that the $\lambda$-operations on cyclic homology appear by integration along the fibers for an epicyclic module. One should obtain in this way a system of coefficients on $\hat{N}^X$.

To an arithmetic variety $X$ should correspond an object in the derived category of epicyclic modules. It should be obtained from the scheme $X$ using for each $p$ the Frobenius in characteristic $p$. 


L. Hesselholt, finite places

Theorem A. Let $k$ be a finite field of order $q = p^r$, let $W$ be its ring of $p$-typical Witt vectors, and let $\iota: W \to \mathbb{C}$ be a choice of embedding. If $f: X \to \text{Spec}(k)$ is a smooth and proper morphism of schemes, then, as meromorphic functions on $\mathbb{C}$,

$$
\zeta(X, s) = \frac{\det_\infty(s \cdot \text{id} - \Theta | \text{TP}_{od}(X) \otimes_{W, \iota} \mathbb{C})}{\det_\infty(s \cdot \text{id} - \Theta | \text{TP}_{ev}(X) \otimes_{W, \iota} \mathbb{C})},
$$

where $\Theta$ is a $\mathbb{C}$-linear graded derivation such that $q^\Theta = \text{Fr}_q^*$ and $\Theta(v) = \frac{2\pi i}{\log q} \cdot v$.

Still big difficulty because use of embedding $W \to \mathbb{C}$ which is a “chimera”.
RH equivalent

RH problem is equivalent to an inequality for real valued functions $f$ on $\mathbb{R}^*_+$ of the form

$$\text{RH} \iff s(f, f) \leq 0, \quad \forall f \mid \int f(u)d^*u = \int f(u)du = 0.$$

$$s(f, g) := N(f \ast \tilde{g}), \quad \tilde{g}(u) := u^{-1}g(u^{-1})$$

$\ast = \text{convolution product on } \mathbb{R}^*_+$

$$N(h) := \sum_{n=1}^{\infty} \Lambda(n)h(n) + \int_{1}^{\infty} \frac{u^2h(u) - h(1)}{u^2 - 1}d^*u + ch(1)$$

$$c = \frac{1}{2}(\log \pi + \gamma).$$
Explicit Formula

\[ F : [1, \infty) \to \mathbb{R} \text{ continuous and continuously differentiable except for finitely many points at which both } F(u) \text{ and } F'(u) \text{ have at most a discontinuity of the first kind, and s.t. for some } \epsilon > 0 : F(u) = O(u^{-1/2-\epsilon}) \]

\[ \Phi(s) = \int_1^\infty F(u) u^{s-1} du \]

\[ \Phi\left(\frac{1}{2}\right) + \Phi\left(-\frac{1}{2}\right) - \sum_{\rho \in \text{Zeros}} \Phi\left(\rho - \frac{1}{2}\right) = \sum_p \sum_{m=1}^\infty \log p \ p^{-m/2} F(p^m) + \]

\[ + \left( \frac{\gamma}{2} + \frac{\log \pi}{2} \right) F(1) + \int_1^\infty \frac{t^{3/2} F(t) - F(1)}{t(t^2 - 1)} dt \]

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Weil’s formulation

$h \in S(C_\mathbb{K})$ a Schwartz function with compact support:

$$\hat{h}(0) + \hat{h}(1) - \sum_{\chi \in \hat{C}_\mathbb{K}, 1} \sum \hat{h}(\tilde{\chi}, \rho) = \sum_v \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|} d^*u$$

where the principal value $\int'_{K_v^*}$ is normalized by the additive character $\alpha_v$ and for any character $\omega$ of $C_\mathbb{K}$

$$\hat{h}(\omega, z) := \int h(u) \omega(u) |u|^z d^*u, \quad \hat{h}(t) := \hat{h}(1, t)$$
The adele class space
and the explicit formulas

Let $\mathbb{K}$ be a global field, the adele class space of $\mathbb{K}$ is the quotient $X_\mathbb{K} = \mathbb{A}_\mathbb{K}/\mathbb{K}^\times$ of the adeles of $\mathbb{K}$ by the action of $\mathbb{K}^\times$ by multiplication.

$$T_\xi(x) := \xi(ux) = \int k(x, y)\xi(y)dy$$

$$k(x, y) = \delta(ux - y),$$

$$\text{Tr}_{\text{distr}}(T') := \int k(x, x)dx = \int \delta(ux - x)dx$$

$$= \frac{1}{|u-1|} \int \delta(z)dz = \frac{1}{|u-1|}$$
Continuous spectrum

Emission spectrum

Absorption spectrum
Critical zeros as absorption spectrum

The spectral side now involves all non-trivial zeros and the geometric side is given by:

\[ \text{Tr}_{\text{distr}} \left( \int h(w)\vartheta(w)d^*w \right) = \sum_v \int_{\mathbb{K}_v^\times} \frac{h(w^{-1})}{|1-w|} d^*w \]

The limit $q \to 1$ and the Hasse-Weil formula

C. Soulé: $\zeta_X(s) := \lim_{q \to 1} Z(X, q^{-s})(q - 1)^{N(1)}, \quad s \in \mathbb{R}$

where $Z(X, q^{-s})$ denotes the evaluation at $T = q^{-s}$ of the Hasse-Weil exponential series

$$Z(X, T) := \exp \left( \sum_{r \geq 1} N(q^r) \frac{T^r}{r} \right)$$

For the projective space $\mathbb{P}^n$: $N(q) = 1 + q + \ldots + q^n$

$$\zeta_{\mathbb{P}^n(\mathbb{F}_1)}(s) = \lim_{q \to 1} \frac{(q - 1)^{n+1}}{n+1} \zeta_{\mathbb{P}^n(\mathbb{F}_q)}(s) = \frac{1}{\prod_0^n (s - k)}$$
The limit $q \to 1$

The Riemann sums of an integral appear from the right hand side:

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\int_1^\infty N(u) u^{-s} d^*u$$

Thus the integral equation produces a precise equation for the **counting function** $N_C(q) = N(q)$ associated to the hypothetical curve $C$:

$$\frac{\partial_s \zeta_Q(s)}{\zeta_Q(s)} = -\int_1^\infty N(u) u^{-s} d^*u$$
The distribution $N(u)$

This equation admits a solution which is a distribution and is given with $\varphi(u) := \sum_{n<u} n \Lambda(n)$, by the equality

$$N(u) = \frac{d}{du} \varphi(u) + \kappa(u)$$

where $\kappa(u)$ is the distribution which appears in the explicit formula

$$\int_{1}^{\infty} \kappa(u)f(u)d^*u = \int_{1}^{\infty} \frac{u^2f(u) - f(1)}{u^2 - 1}d^*u + cf(1), \quad c = \frac{1}{2}(\log \pi + \gamma)$$

The conclusion is that the distribution $N(u)$ is positive on $(1, \infty)$ and is given by

$$N(u) = u - \frac{d}{du} \left( \sum_{\rho \in \mathbb{Z}} \text{order}(\rho) \frac{u^\rho + 1}{\rho + 1} \right) + 1$$
$$J(u) = \int N(u) \, d\, u$$

$$J_m(u)$$
Space \( X_Q := \mathbb{Q}^\times \backslash \text{A}_Q / \mathbb{Z}^\times \)

The quotient \( X_Q := \mathbb{Q}^\times \backslash \text{A}_Q / \mathbb{Z}^\times \) of the adele class space \( \mathbb{Q}^\times \backslash \text{A}_Q \) of the rational numbers by the maximal compact subgroup \( \mathbb{Z}^\times \) of the idele class group, gives by considering the induced action of \( \mathbb{R}^\times_+ \), the above counting distribution \( N(u), u \in [1, \infty) \), which determines, using the Hasse-Weil formula in the limit \( q \to 1 \), the complete Riemann zeta function.
Automorphisms of $\mathbb{R}^{\text{max}}_+$

$\text{Fr}_\lambda(x) = x^\lambda$ automorphisms of $\mathbb{R}^{\text{max}}_+$.

$\text{Gal}_B(\mathbb{R}^{\text{max}}_+) = \mathbb{R}_+^\times$
Points of the arithmetic site

over $\mathbb{R}^{\text{max}}_+$

These are defined as pairs $(p, f_p^\#)$ of a point $p$ of $\hat{\mathbb{N}}^\times$ and local morphism $f_p^\# : \mathcal{O}_p \to \mathbb{R}^{\text{max}}_+$.

Theorem

The points $\mathcal{A}(\mathbb{R}^{\text{max}}_+)$ of $(\hat{\mathbb{N}}^\times, \mathbb{Z}^{\text{max}})$ on $\mathbb{R}^{\text{max}}_+$ form the double quotient $\mathbb{Q}^\times \backslash \mathbb{A}_Q / \hat{\mathbb{Z}}^*$. The action of the Frobenius $\text{Fr}_\lambda$ of $\mathbb{R}^{\text{max}}_+$ corresponds to the action of the idele class group.
| $C$ curve defined over $\mathbb{F}_q$ | **Arithmetic Site** 
$\mathcal{A} = (\hat{\mathbb{N}}, \mathbb{Z}_{\text{max}})$ over $\mathcal{B}$ |
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<td>Galois on $C(\overline{\mathbb{F}}_q)$</td>
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| $\Psi$ Frobenius Correspondence | **Correspondences** $\Psi(\lambda)$  
$\lambda \in \mathbb{R}^*_+$ on $\mathcal{A} \times \mathcal{A}$ |
Frobenius Correspondences
Theorem

Let $\lambda, \lambda' \in \mathbb{R}_+^*$ with $\lambda \lambda' \notin \mathbb{Q}$. The composition

$$\psi(\lambda) \circ \psi(\lambda') = \psi(\lambda \lambda')$$

Same if $\lambda$ and $\lambda'$ are rational. If $\lambda \notin \mathbb{Q}, \lambda' \notin \mathbb{Q}$ and $\lambda \lambda' \in \mathbb{Q}$,

$$\psi(\lambda) \circ \psi(\lambda') = \psi(\lambda \lambda') \circ \text{Id}_\epsilon = \text{Id}_\epsilon \circ \psi(\lambda \lambda')$$

where $\text{Id}_\epsilon$ is the tangential deformation of $\text{Id}$. 

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Divisors and intersection

Intersection $D \bullet D'$ of formal divisors

$$D := \int h(\lambda) \psi_\lambda d^* \lambda$$

$$D \bullet D' := < D \star \tilde{D}', \Delta >$$

where $\tilde{D}'$ is the transposed $D'$ and composition $D \star \tilde{D}'$ is bilinear $< D \star \tilde{D}', \Delta >$ using the distribution $N(u)$ and correspondence $\psi_\lambda$ of degree $\lambda$. 
Negativity ⇐⇒ RH

- Horizontal and vertical $\xi_j$.

- RH is equivalent to inequality

$$D \cdot D \leq 2(D \cdot \xi_1)(D \cdot \xi_2)$$

Incompatibility of $\leq$ with naive positivity resolved by small lemma (cf Matuck-Tate-Grothendieck)
Extension of scalars to $\mathbb{R}_{\text{max}}$

The following holds:

$$\mathbb{Z}_{\text{max}} \hat{\otimes}_{B} \mathbb{R}_{\text{max}} \cong \mathcal{R}(\mathbb{Z})$$

$\mathcal{R}(\mathbb{Z}) = \text{semiring of continuous, convex, piecewise affine functions on } \mathbb{R}_{+} \text{ with slopes in } \mathbb{Z} \subset \mathbb{R} \text{ and only finitely many discontinuities of the derivative}$

These functions are endowed with the pointwise operations of functions with values in $\mathbb{R}_{\text{max}}$
Points of the topos $[0, \infty) \times \mathbb{N}^\times$

**Theorem**: The points of the topos $[0, \infty) \times \mathbb{N}^\times$ form the double quotient $\mathbb{Q}^\times \backslash \mathbb{A}_Q / \hat{\mathbb{Z}}^*$. 

**Corollary**: One has a canonical isomorphism $\Theta$ between the points of the topos $[0, \infty) \times \mathbb{N}^\times$ and $\mathcal{A}(\mathbb{R}^\text{max})$ i.e. the points of the arithmetic site defined over $\mathbb{R}^\text{max}$. 

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**Structure sheaf of** \([0, \infty) \ltimes \mathbb{N}^\times\)**

This is the sheaf on \([0, \infty) \ltimes \mathbb{N}^\times\) associated to convex, piecewise affine functions with integral slopes.

Same as for the localization of zeros of analytic functions

\(f(X) = \sum a_n X^n\) in an annulus

\[
A(r_1, r_2) = \{ z \in K \mid r_1 < |z| < r_2 \}
\]

\[
\tau(f)(x) := \max_n \{-nx - v(a_n)\}, \quad \forall x \in (-\log r_2, -\log r_1)
\]

\[
\tau(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{-x+i\theta})| d\theta
\]
\[
\tilde{C} = C \otimes_{\overline{F}_q} \overline{F}_q
\]
on \overline{F}_q

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<th>Structure sheaf of ( \tilde{C} )</th>
<th>Sheaf of fractions</th>
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<td>( O_{\tilde{C}} = C \otimes_{\overline{F}_q} \overline{F}_q )</td>
<td>( \mathbb{Z}<em>{\max} \otimes</em>{\mathbb{B}} \mathbb{R}_+^{\max} \rightarrow ) Sheaf of convex piecewise affine functions, slopes ( \in \mathbb{Z} )</td>
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Periodic Orbits

By restriction of the structure sheaf of

\[ \hat{A} = ([0, \infty) \times \mathbb{N}^X, \mathcal{O}) \]

to periodic orbits (i.e. the image of Spec \( \mathbb{Z} \)) one obtains, for each prime \( p \) a real analogue

\[ C_p = \mathbb{R}_+^* / p^\mathbb{Z} \]

of Jacobi elliptic curve \( \mathbb{C}^* / q^\mathbb{Z} \).
| Elliptic curve over $\mathbb{C}$ | **Periodic orbit**  
| Curve $C_p$ over $\mathbb{R}_+^{\text{max}}$ |
|---|---|
| Points over $\mathbb{C}$: $\mathbb{C}^\times/q\mathbb{Z}$ | $\mathbb{R}_+/p\mathbb{Z}, \, H \subset \mathbb{R}, \, H \sim H_p$ |
| Structure sheaf periodic functions $f(qz) = f(z)$ | Sheaf of periodic convex piecewise affine functions, **slopes** $\in H_p$ |
| Sheaf $\mathcal{K}$ of rational functions $f(qz) = f(z)$ | Sheaf of periodic $f(p\lambda) = f(\lambda)$ continuous piecewise affine functions, **slopes** $\in H_p$ |
For $W \subset C_p$ open, $\mathcal{O}_p(W)$ is simplifiable, one lets $\mathcal{K}_p$ the sheaf associated to the presheaf $W \mapsto \text{Frac} \mathcal{O}_p(W)$.

**Lemma** The sections of the sheaf $\mathcal{K}_p$ are continuous piecewise affine functions with slopes in $H_p$ endowed with max ($\vee$) and the sum.

$$(x - y) \vee (z - t) = ((x + t) \vee (y + z)) - (y + t).$$
**Cartier divisors**

**Lemma**: The sheaf $\text{CDiv}(C_p)$ of Cartier divisors i.e. the quotient sheaf $K_p^\times / \mathcal{O}_p^\times$, is isomorphic to the sheaf of naive divisors $H \mapsto D(H) \in H$,

$$\forall \lambda, \exists V \text{ open } \lambda \in V, \ D(\mu) = 0, \quad \forall \mu \in V, \ \mu \neq \lambda$$

Point $p_H$ associated to $H \subset \mathbb{R}$ and $f$ section of $\mathcal{K}$ at $p_H$.

$$\text{Order}(f) = h_+ - h_- \in H \subset \mathbb{R}$$

$$h_{\pm} = \lim_{\epsilon \to 0^\pm} \frac{f((1 + \epsilon)H) - f(H)}{\epsilon}.$$
**Divisors**

**Definition**: A divisor is a global section of \( \mathcal{K}_p^\times / \mathcal{O}_p^\times \), i.e., a map \( H \to D(H) \in H \) vanishing except on finitely many points.

**Proposition**: (i) The divisors \( \text{Div}(C_p) \) form an abelian group under addition.

(ii) The condition \( D'(H) \geq D(H), \forall H \in C_p \), defines a partial order on \( \text{Div}(C_p) \).

(iii) The **degree** map is additive and order preserving

\[
\text{deg}(D) := \sum D(H) \in \mathbb{R}.
\]
The sheaf $\mathcal{K}_p$ admits global sections:

$$\mathcal{K} := \mathcal{K}(C_p) = H^0(\mathbb{R}^*/p\mathbb{Z}, \mathcal{K}_p)$$

the semifield of global sections.

**Principal divisors** : The map which to $f \in \mathcal{K}^\times$ associates the divisor

$$(f) := \sum_H (H, \text{Ord}_H(f)) \in \text{Div}(C_p)$$

is a group morphism $\mathcal{K}^\times \to \mathcal{P} \subset \text{Div}(C_p)$.

The subgroup $\mathcal{P} \subset \text{Div}(C_p)$ of principal divisors is **contained in the kernel** of the morphism $\text{deg} : \text{Div}(C_p) \to \mathbb{R}$:

$$\sum_H \text{Ord}_H(f) = 0, \quad \forall f \in \mathcal{K}^\times.$$
For $p > 2$ one considers the ideal $(p - 1)H_p \subset H_p$.

$$0 \to (p - 1)H_p \to H_p \overset{r}{\to} \mathbb{Z}/(p - 1)\mathbb{Z} \to 0$$

**Lemma**: For $H \subset \mathbb{R}$, $H \cong H_p$, the map $\chi : H \to \mathbb{Z}/(p - 1)\mathbb{Z}$, $\chi(\mu) = r(\mu/\lambda)$ where $H = \lambda H_p$ is independent of the choice of $\lambda$.

**Theorem**

The map $(\text{deg}, \chi)$ is a **group isomorphism**

$$(\text{deg}, \chi) : \text{Div}(C_p)/\mathcal{P} \to \mathbb{R} \times (\mathbb{Z}/(p - 1)\mathbb{Z})$$

where $\mathcal{P}$ is the subgroup of principal divisors.
Theta Functions on $C_p = \mathbb{R}^*/p^\mathbb{Z}$

\[ \prod_0^\infty (1 - t^m w) \rightarrow f_+(\lambda) := \sum_0^\infty (0 \lor (1 - p^m \lambda)) \]

\[ \prod_1^\infty (1 - t^m w^{-1}) \rightarrow f_-(\lambda) := \sum_1^\infty (0 \lor (p^{-m} \lambda - 1)) \]

**Theorem**

Any $f \in \mathcal{K}(C_p)$ has a canonical decomposition

\[ f(\lambda) = \sum_i \Theta_{h_i, \mu_i}(\lambda) - \sum_j \Theta_{h'_j, \mu'_j}(\lambda) - h \lambda + c \]

where $c \in \mathbb{R}$, $(p - 1)h = \sum h_i - \sum h'_j$ and $h_i \leq \mu_i < ph_i$, $h'_j \leq \mu'_j < ph'_j$. 

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Definition: Let $D \in \text{Div}(C_p)$ one lets

$$H^0(D) := \{ f \in \mathcal{K}(C_p) \mid D + (f) \geq 0 \}$$

It is an $\mathbb{R}_{\max}$-module, $f, g \in H^0(D) \Rightarrow f \vee g \in H^0(D)$.

Lemma: Let $D \in \text{Div}(C_p)$ be a divisor, one gets a filtration of $H^0(D)$ by $\mathbb{R}_{\max}$-sub-modules:

$$H^0(D)_{\rho} := \{ f \in H^0(D) \mid \| f \|_p \leq \rho \}$$

using the $p$-adic norm.
Real valued Dimension

\[ \text{Dim}_\mathbb{R}(H^0(D)) := \lim_{n \to \infty} p^{-n} \dim_{\text{top}}(H^0(D)p^n) \]

where the topological dimension \( \dim_{\text{top}}(X) \) is the number of real parameters on which solutions depend.

Riemann-Roch Theorem

(i) Let \( D \in \text{Div}(C_p) \) a divisor with \( \deg(D) \geq 0 \), then

\[ \lim_{n \to \infty} p^{-n} \dim_{\text{top}}(H^0(D)p^n) = \deg(D) \]

(ii) One has the Riemann-Roch formula :

\[ \text{Dim}_\mathbb{R}(H^0(D)) - \text{Dim}_\mathbb{R}(H^0(-D)) = \deg(D), \quad \forall D \in \text{Div}(C_p). \]
Back to the goal: RR on the square

Integrals of Frobenius correspondences

\[ D := \int h(\lambda) \psi_\lambda d^* \lambda \]

One needs a Riemann-Roch formula

\[ \dim H^0 - \dim H^1 + \dim H^2 = \frac{1}{2} D \bullet D \]

in order to make \( \pm D \) effective and get a contradiction

(Negativity \( \iff \) RH)

Open problem: suitable definition of \( H^1 \)
Tropical RR theorem

Baker, Norine, Gathmann, Kerber.

The power in these results is the existence part, it uses

Game theory, Potential theory

**but** the definition of the terms in the RR formula are **not** given in terms of the dimension of $H^0$!

(counter-example of Yoshitomi)
The new development in our strategy is to deduce the existence part of the Riemann-Roch formula in the tropical shadow (i.e. on the square of the Scaling Site) from a corresponding formula holding on the analytic geometric version of the space (i.e. its complex lift).

The advantage of working in characteristic zero is to have already available all the algebraic and analytical tools needed to prove such result.
Jensen

$f(z)$ holomorphic function in an annulus

$$A(r_1, r_2) = \{ z \in \mathbb{C} \mid r_1 < |z| < r_2 \}$$

$$\tau(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{-x+i\theta})| d\theta.$$ 

$$\exists z \mid f(z) = 0, \ - \log |z| = x \iff \Delta(\tau(f))(x) \neq 0$$

Tropical zeros of $\tau(f)$ are the $-\log |z|$.

Tropical descent

$$D + (\tau) := \sum n_j \delta_{\lambda_j} + \Delta(\tau) \geq 0.$$
First attempt: punctured disk

\[ \mathbb{D}^* \times \mathbb{N}^\times \rightarrow [0, \infty) \times \mathbb{N}^\times \]

\[ \mathbb{D}^* := \{ q \in \mathbb{C} \mid 0 < |q| \leq 1 \} \]

The monoid \( \mathbb{N}^\times \) acts naturally on \( \mathbb{D}^* \) by means of the map \( q \mapsto q^n \). In this way, one defines a ringed topos by endowing the topos \( \mathbb{D}^* \times \mathbb{N}^\times \) with the structure sheaf \( \mathcal{O} \) of complex analytic functions.

The map

\[ \mathbb{D}^* \ni q \mapsto - \log |q| \in [0, \infty) \]

extends to a geometric morphism of toposes \( \mathbb{D}^* \times \mathbb{N}^\times \rightarrow [0, \infty) \times \mathbb{N}^\times \).
Almost periodic analytic fcts

In order to lift divisors of the form $D(f) = \int f(\lambda)\delta_{\lambda} d^*\lambda$ to a discrete divisor $\tilde{D}(f)$ on a complex geometric space, one uses the Jessen theory of analytic almost periodic functions

$$\varphi(\sigma) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \log |f(\sigma + it)| dt$$  \hspace{1cm} (1)

$$\lim_{T \to \infty} \frac{N(T)}{2T} = \frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{2\pi}.\hspace{1cm} (2)$$
Proetale cover $\tilde{D}^* \times \mathbb{N}^\times$

$$\tilde{D}^* := \lim_{\leftarrow} \mathbb{N}^\times(D^*, z \mapsto z^n).$$

One uses: Witt construction in characteristic 1 and Teichmuller lift $[-]: \mathbb{R}_+^{\text{max}} \to W$, to define

$$q(z) := [\lvert z \rvert] \exp(2\pi i \text{arg} z)$$

The structure sheaf of the pro-étale cover involves the ring $W[q^r]$ generated by rational powers $q^r$ of $q$ over $W$.

**Compare to perfectoid torus.**
Adelic description : $\mathcal{C}_Q$

Compactification $G := \varprojlim_{N \times} \mathbb{R}/n\mathbb{Z}$,

$$\mathcal{C}_Q = \mathbb{Q}^* \backslash (\mathbb{A}_Q \times G) = P(\mathbb{Q}) \backslash \overline{P(\mathbb{A}_Q)}.$$  \hspace{1cm} (3)

The obtained noncommutative space is the moduli space of elliptic curves endowed with a triangular structure, up to isogenies.

A **triangular structure** on an elliptic curve $E$ is a pair $(\xi, \eta)$ of elements of the Tate module $T(E)$, such that $\xi \neq 0$ and $<\xi^\perp, \eta> = \mathbb{Z}$.

$\xi^\perp := \{ \chi \in \text{Hom}(E, \mathbb{R}/\mathbb{Z}) \mid T(\chi)(\xi) = 0 \} \subset \text{Hom}(E, \mathbb{R}/\mathbb{Z})$

The space $\mathcal{C}_Q$ has a foliation ! of complex dimension 1 and an additional real deformation parameter.
$C_Q$ and the $GL(2)$-system

We relate the noncommutative space $C_Q = \overline{P(Q)\backslash P(A_Q)}$ to the $GL(2)$-system. This system was conceived as a higher dimensional generalization of the BC-system and its main new feature is provided by its arithmetic subalgebra of modular functions. The classical Shimura scheme $Sh(GL_2, \mathbb{H}^\pm) := GL_2(Q)\backslash GL_2(A_Q)/\mathbb{C}^\times$ appears as the set of classical points of the noncommutative space $\overline{Sh^{nc}(GL_2, \mathbb{H}^\pm)}$ underlying the $GL(2)$-system. This noncommutative space admits a simple description as the double quotient

$$\overline{Sh^{nc}(GL_2, \mathbb{H}^\pm)} = GL_2(Q)\backslash M_2(A_Q)^\bullet/\mathbb{C}^\times$$

obtained by replacing in the construction of $Sh(GL_2, \mathbb{H}^\pm)$ the middle term $GL_2(A_Q)$ by $M_2(A_Q)^\bullet := M_2(A_Q, f) \times$
\((M_2(\mathbb{R}) \setminus \{0\})\) i.e. the space of matrices (of adèles) with non-zero archimedean component.
By definition $Sh(\text{GL}_2, \mathbb{H}^\pm)$ is the quotient

$$Sh(\text{GL}_2, \mathbb{H}^\pm) := \text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}_\mathbb{Q}) / \mathbb{C}^\times$$

$$= \text{GL}_2(\mathbb{Q}) \backslash (\text{GL}_2(\mathbb{A}_\mathbb{Q}, f) \times \mathbb{H}^\pm),$$

where the left action of $\text{GL}_2(\mathbb{Q})$ in $(\text{GL}_2(\mathbb{A}_\mathbb{Q}, f) \times \mathbb{H}^\pm)$ is via the diagonal embedding in the product $\text{GL}_2(\mathbb{A}_\mathbb{Q}, f) \times \text{GL}_2(\mathbb{R})$. 
$Sh(\text{GL}_2, \mathbb{H}^{\pm})$

$Sh(\text{GL}_2, \mathbb{H}^{\pm})$ is a scheme over $\mathbb{C}$ which is the inverse limit of the Shimura varieties obtained as quotients by compact open subgroups $K \subset \text{GL}_2(\mathbb{A}_{\mathbb{Q}}, f)$. The space $Sh(\text{GL}_2, \mathbb{H}^{\pm})$ has infinitely many connected components. They are the fibers of the map

$$\text{det} \times \text{sign} : Sh(\text{GL}_2, \mathbb{H}^{\pm}) \to Sh(\text{GL}_1, \{\pm 1\}), \quad (4)$$

where the determinant $\text{det} : \text{GL}_2(\mathbb{A}_{\mathbb{Q}}, f) \to \text{GL}_1(\mathbb{A}_{\mathbb{Q}}, f)$ gives a map to the group of finite ideles. Passing to the quotient gives a map to the idele class group modulo its archimedean component, i.e. here the group $\hat{\mathbb{Z}}^\times$.

The fiber of the map (4) over the point $(1, 1) \in Sh(\text{GL}_1, \{\pm 1\})$ is the connected quotient

$$Sh_0(\text{GL}_2, \mathbb{H}^{\pm}) := \text{SL}_2(\mathbb{Q}) \backslash (\text{SL}_2(\mathbb{A}_{\mathbb{Q}}, f) \times \mathbb{H}).$$
Let $Y(N) = \Gamma(N) \backslash \mathbb{H}$ be the modular curve of level $N$, where $\Gamma(N)$ is the principal congruence subgroup of $\Gamma = \text{SL}_2(\mathbb{Z})$. One has

$$Sh_0(\text{GL}_2, \mathbb{H}^\pm) = \underleftarrow{\lim}_ N \Gamma(N) \backslash \mathbb{H} = \underleftarrow{\lim}_ N Y(N).$$
The noncommutative space underlying the GL(2)-system contains the quotient

\[
Sh_{\text{nc}}(\text{GL}_2, \mathbb{H}^{\pm}) := \text{GL}_2(\mathbb{Q}) \backslash (M_2(A_{\mathbb{Q}, f}) \times \mathbb{H}^{\pm}),
\]
and enlarges it by taking cusps into account. It is defined as the double quotient

\[
\overline{Sh_{\text{nc}}(\text{GL}_2, \mathbb{H}^{\pm})} := \text{GL}_2(\mathbb{Q}) \backslash M_2(A_{\mathbb{Q}})^\bullet / \mathbb{C}^\times,
\]
where one sets

\[
M_2(A_{\mathbb{Q}})^\bullet := M_2(A_{\mathbb{Q}, f}) \times (M_2(\mathbb{R}) \setminus \{0\}).
\]
Relation of $C_Q = P(Q) \setminus P(A_Q)$ with the GL(2)-system.

(i) The inclusion $\overline{P(\mathbb{R})} \subset (M_2(\mathbb{R}) \setminus \{0\})$ induces a bijective of $\overline{P(\mathbb{R})}$ with the complement in $(M_2(\mathbb{R}) \setminus \{0\}) / \mathbb{C}^\times$ of the point $\infty$ given by the class of matrices with vanishing second line.

(ii) The inclusion $\overline{P(A_Q)} \subset M_2(A_Q)^\bullet$ induces a morphism of noncommutative spaces

$$C_Q = P(Q) \setminus P(A_Q) \xrightarrow{\theta} \text{Sh}^{nc}(\text{GL}_2, \mathbb{H}^\pm).$$

(7)
Surjectivity (archimedean place)

For real matrices the following implication holds
\[
\begin{pmatrix}
  a & b \\
  0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
  a' & b' \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  x & y \\
  -y & x
\end{pmatrix}
\implies
a = a' \& b = b'.
\]

Thus the induced map \( \overline{P(\mathbb{R})} \to (M_2(\mathbb{R}) \setminus \{0\})/\mathbb{C}^\times \) is injective.

Also, and again for real matrices one has, provided \( c \) or \( d \) is non-zero
\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
= 
\begin{pmatrix}
  \frac{ad-bc}{c^2+d^2} & \frac{ac+bd}{c^2+d^2} \\
  0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
  d & -c \\
  c & d
\end{pmatrix}.
\]  \( (8) \)

Thus all matrices in \( M_2(\mathbb{R}) \) whose second line is non-zero belong to \( \overline{P(\mathbb{R})}/\mathbb{C}^\times \). When both \( c \) and \( d \) are zero,
one derives

\[
\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.
\]

This means that when \( c = d = 0 \) the right action of \( \mathbb{C}^\times \) determines a single orbit \( \{\infty\} \) provided one stays away from the matrix 0. Thus one obtains a canonical bijection

\[
(M_2(\mathbb{R}) \setminus \{0\}) / \mathbb{C}^\times = \overline{P(\mathbb{R})} \cup \{\infty\}.
\]
**Q-lattices**

A two dimensional $\mathbb{Q}$-lattice is a pair $(\Lambda, \phi)$ where $\Lambda \subset \mathbb{C}$ is a lattice and $\phi : \mathbb{Q}^2/\mathbb{Z}^2 \to \mathbb{Q}\Lambda/\Lambda$ is an arbitrary morphism of abelian groups. The morphism $\phi$ encodes the non-archimedean components of the lattice.

The set of 2-dimensional $\mathbb{Q}$-lattices is the quotient space

$$\Gamma \backslash (M_2(\hat{\mathbb{Z}}) \times \text{GL}_2^+(\mathbb{R})),$$

where $\Gamma = \text{SL}_2(\mathbb{Z})$. 

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We use the basis \( \{e_1 = 1, e_2 = -i\} \) of \( \mathbb{C} \) as a 2-dimensional \( \mathbb{R} \)-vector space to let \( \text{GL}_2(\mathbb{R}) \) act on \( \mathbb{C} \) as \( \mathbb{R} \)-linear transformations. More precisely,

\[
\alpha(xe_1 + ye_2) = (ax+by)e_1 + (cx+dy)e_2, \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{R}).
\]

Every 2-dimensional \( \mathbb{Q} \)-lattice \( (\Lambda, \phi) \) can then be described by the data

\[
(\Lambda, \phi) = (\alpha^{-1}\Lambda_0, \alpha^{-1}\rho), \quad \Lambda_0 := \mathbb{Z}e_1 + \mathbb{Z}e_2 = \mathbb{Z} + i\mathbb{Z}
\]

for some \( \alpha \in \text{GL}_2^+(\mathbb{R}) \) and some \( \rho \in M_2(\hat{\mathbb{Z}}) \) unique up to the left diagonal action of \( \Gamma = \text{SL}_2(\mathbb{Z}) \).

\( \text{Q-lattices} \)
The action of $\mathbb{C}^\times$ by scaling on $\mathbb{Q}$-lattices is given by

$$\lambda(\Lambda, \phi) = (\lambda \Lambda, \lambda \phi), \quad \forall \lambda \in \mathbb{C}^\times.$$  \hspace{1cm} (9)

The set of 2-dimensional $\mathbb{Q}$-lattices up to scaling is identified with

$$\Gamma \backslash (M_2(\hat{\mathbb{Z}}) \times \text{GL}_2^+(\mathbb{R}))/\mathbb{C}^\times = \Gamma \backslash (M_2(\hat{\mathbb{Z}}) \times \mathbb{H}).$$  \hspace{1cm} (10)
$C_Q$ and parabolic $\mathbb{Q}$-lattices

We give a geometric interpretation of the subspace

$$C_Q^0 := P(\mathbb{Q}) \setminus (\overline{P(A_Q, f)} \times P(\mathbb{R})) \subset P(\mathbb{Q}) \setminus \overline{P(A_Q)} =: C_Q.$$
Let $\Lambda = \alpha^{-1}\Lambda_0$ be a $\mathbb{Q}$-lattice, with $\alpha \in P(\mathbb{R})$. Then

(i) $\Im (\Lambda) = \mathbb{Z}$.

(ii) The linear map $\Im$ induces a group homomorphism $\Im : E \to \mathbb{R}/\mathbb{Z}$ from the elliptic curve $E = \mathbb{C}/\Lambda$ to the abelian group $U(1) := \mathbb{R}/\mathbb{Z}$, i.e. a character of the abelian group $E$.

(iii) The orthogonal lattice $\Lambda^\perp$ contains the vector $e_2 = -i$. 

**Parabolic $\mathbb{Q}$-lattices**
Parabolic $\mathbb{Q}$-lattices

Let $(\Lambda, \phi)$ be a two dimensional $\mathbb{Q}$-lattice described by data $(\Lambda, \phi) = (\alpha^{-1} \Lambda_0, \alpha^{-1} \rho)$, for some $\alpha \in \text{GL}_2^+(\mathbb{R})$ and some $\rho \in M_2(\hat{\mathbb{Z}})$. Then,

$$(\rho, \alpha) \in \Gamma \setminus \left( \overline{P(\hat{\mathbb{Z}})} \times P^+(\mathbb{R}) \right) \iff \Im(\Lambda) = \mathbb{Z} \land \chi \circ \phi(u) = y \quad \forall u = (x, y) \in \mathbb{Q}^2 / \mathbb{Z}^2$$

where $\chi : \mathbb{C} / \Lambda \to \mathbb{R} / \mathbb{Z}$ is given by $\chi = -\Im$ and $\Gamma = \text{SL}_2(\mathbb{Z})$ acts diagonally.
A *parabolic* $\mathbb{Q}$-lattice is a $\mathbb{Q}$-lattice of the form $(\Lambda, \phi) = (\alpha^{-1}\Lambda_0, \alpha^{-1}\rho)$, where $\rho \in \overline{P(\mathbb{Z})}$ and $\alpha \in P^+(\mathbb{R})$.

We say that a parabolic $\mathbb{Q}$-lattice $(\Lambda, \phi)$ is degenerate when $\rho_1 = 0$. 

**Parabolic $\mathbb{Q}$-lattices**
Commensurability of $\mathbb{Q}$-lattices

Two $\mathbb{Q}$-lattices are said to be commensurable

$$(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$$

when

$$\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2 \quad \text{and} \quad \phi_1 = \phi_2 \text{ semimodule.} \quad \Lambda_1 + \Lambda_2. \quad (11)$$

Commensurability is an equivalence relation, the space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices up to scaling is given by the quotient space

$$\text{GL}_2^+(\mathbb{Q}) \backslash (M_2(\mathbb{A}_\mathbb{Q}, f) \times \mathbb{H}) \quad (12)$$
Theorem

(i) Two parabolic two dimensional $\mathbb{Q}$-lattices $(\Lambda_j, \phi_j) = (\alpha_j^{-1} \Lambda_0, \alpha_j^{-1} \rho_j)$, $j = 1, 2$, with $\rho_j \in P(\hat{\mathbb{Z}})$ and $\alpha_j \in P^+(\mathbb{R})$ are commensurable (as $\mathbb{Q}$-lattices) if and only if there exists $g \in P^+(\mathbb{Q})$ such that $\rho_2 = g \rho_1$ and $\alpha_2 = g \alpha_1$.

(ii) The space of parabolic $\mathbb{Q}$-lattices up to commensurability is canonically isomorphic to the quotient $C_Q^0$. 
Link with BC

From the point of view of noncommutative geometry, the quotient space derived by applying the commensurability relation on the space of parabolic $\mathbb{Q}$-lattices is best described by considering the crossed product by the Hecke algebra of double classes of the subgroup $P^+(\mathbb{Z}) \subset P^+(\mathbb{Q})$. We find quite remarkable (and encouraging) that this Hecke algebra is precisely the one on which the BC-system is based.
We think of the global Tate module as the abelian group

\[ TE = \text{Hom}(\mathbb{Q}/\mathbb{Z}, E_{\text{tor}}). \]

We denote by \( E_{\text{tor}} = \mathbb{Q}\Lambda/\Lambda \) the torsion subgroup of the elliptic curve \( E \).
**Lemma**

Let \((\Lambda, \phi)\) be a two dimensional \(\mathbb{Q}\)-lattice, \(E = \mathbb{C}/\Lambda\) the associated elliptic curve, and \((\xi, \eta)\) the related pair of points in the total Tate module \(TE\). Then \((\Lambda, \phi)\) is a parabolic \(\mathbb{Q}\)-lattice if and only if

\[
\Im(\Lambda) = \mathbb{Z} \quad \& \quad T(\chi)(\xi) = 0, \quad T(\chi)(\eta) = \text{Id}, \quad \text{for } \chi = -\Im. \tag{13}
\]
Up to scale

Passing from a parabolic $\mathbb{Q}$-lattice to the associated triple $(E; \xi, \eta)$ is equivalent to assign the map $\theta$ from parabolic $\mathbb{Q}$-lattices to $\mathbb{Q}$-lattices up to scale

$$P^+(\mathbb{Z}) \backslash (\overline{P(\hat{\mathbb{Z}}) \times P^+(\mathbb{R})}) \xrightarrow{\theta} \Gamma \backslash (M_2(\hat{\mathbb{Z}}) \times \text{GL}^+_2(\mathbb{R}))/\mathbb{C}^\times.$$
Injectivity of $\theta$

The map $\theta$ is injective except when the parabolic $\mathbb{Q}$-lattices are degenerate. The following formula defines a free action of $\mathbb{Z}$ on the degenerate parabolic $\mathbb{Q}$-lattices, whose orbits are the fibers of the map $\theta$

$$\tau(c)(p, \alpha) = (p, t_c(\alpha)), \quad t_c(\alpha) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \alpha (1+c\bar{z})^{-1}, \quad \forall c \in \mathbb{Z}. $$

Here, $p = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \overline{P(\mathbb{Z})}$, $\alpha = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in P^+(\mathbb{R})$, and $z = x + iy \in \mathbb{C}$. 
**Theorem**

Let $E$ be an elliptic curve together with a pair of elements $(\xi, \eta)$ of the total Tate module $TE$. Assume $\xi \neq 0$. Then the corresponding $\mathbb{Q}$-lattice belongs to the range of the map $\theta$ if and only if one has $\langle \xi^\perp, \eta \rangle = \mathbb{Z}$, where $\langle \xi^\perp, \eta \rangle := \{ T(\chi)(\eta) \mid \chi \in \xi^\perp \} \subset \hat{\mathbb{Z}}$. 
Triangular structure

A triangular structure on an elliptic curve $E$ is a pair $(\xi, \eta)$ of elements of the Tate module $T(E)$, such that $\xi \neq 0$ and $\langle \xi^\perp, \eta \rangle \subseteq \mathbb{Z}$. 
We recall that an isogeny from an abelian variety $A$ to another $B$ is a surjective morphism with finite kernel.

At the geometric level, the commensurability relation is obtained from the following notion of isogeny between triangular elliptic curves

An isogeny $f : (E, \xi, \eta) \to (E', \xi', \eta')$ of triangular elliptic curves is an isogeny $f : E \to E'$ such that $T(f)(\xi) = \xi'$ and $T(f)(\eta) = \eta'$. 

**Isogenies**
Commensurability and isogenies

Let \((E; \xi, \eta)\) and \((E'; \xi', \eta')\) be two triangular elliptic curves and \((\Lambda, \phi)\) and \((\Lambda', \phi')\) the associated parabolic \(\mathbb{Q}\)-lattices.

(i) Let \(f : (E; \xi, \eta) \rightarrow (E'; \xi', \eta')\) be an isogeny of triangular elliptic curves. Then \(\Lambda \subset \Lambda'\), \(f : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'\) is the map induced by the identity and \(\phi' = f \circ \phi\).

(ii) The parabolic \(\mathbb{Q}\)-lattices \((\Lambda, \phi)\) and \((\Lambda', \phi')\) are commensurable if and only if there exist two isogenies \(f : (E, \xi, \eta) \rightarrow (E'', \xi'', \eta'')\) and \(f' : (E', \xi', \eta') \rightarrow (E'', \xi'', \eta'')\) to the same triangular elliptic curve.
The complex structure

The map \( \theta \) from parabolic \( \mathbb{Q} \)-lattices to \( \mathbb{Q} \)-lattices up to scale is injective except in the degenerate case. Thus \( \theta \) provides, by pull back, a large class of functions, by implementing the arithmetic subalgebra of the \( \text{GL}_2 \)-system. The functions in this algebra are holomorphic for the natural complex structure on the moduli space of elliptic curves and we compare this complex structure with the one on the space \( \Pi = P^+(\mathbb{Z}) \backslash (\overline{P(\mathbb{Z})} \times P^+(\mathbb{R})) \) defined using the right action of \( P^+(\mathbb{R}) \).
The complex structure

The natural map from parabolic $\mathbb{Q}$-lattices to $\mathbb{Q}$-lattices up to scale

$$\theta : \Pi \to \Gamma \backslash (M_2(\mathbb{Z}) \times \text{GL}_2^+(\mathbb{R})) / \mathbb{C}^\times$$  \hspace{1cm} (14)

is holomorphic for the canonical complex structure on the moduli space of elliptic curves and the complex structure on $\Pi$ associated to the right action of $P^+(\mathbb{R})$. 
Boundary cases

Degeneracy occurring when for \( \alpha = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in P^+(\mathbb{R}) \), \( a \) tends to 0. We follow the lattice \( \Lambda = \alpha^{-1} \Lambda_0 \) up to scale.

\[
a \alpha^{-1} = a \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix},
\]

\[
a \alpha^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - by \\ ay \end{pmatrix}.
\]

Thus when \( a \to 0 \) the lattice \( \Lambda = \alpha^{-1} \Lambda_0 \) up to scale converges pointwise (i.e. for each fixed pair \((x, y)\)) towards the subgroup of \( \mathbb{R} \subset \mathbb{C} \) given by

\[
\Lambda(b) := \mathbb{Z} + b\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}.
\]
The subgroup $\Lambda(b)$ only depends upon $b \in \mathbb{R}/\mathbb{Z}$ and the quotient $\mathbb{R}/\Lambda(b)$ corresponds to the noncommutative torus $\mathbb{T}_b^2$. In fact, the composition with $\rho = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} \in P(\hat{\mathbb{Z}})$ gives the following $\mathbb{Q}$-pseudolattice:

$$\phi : \mathbb{Q}^2/\mathbb{Z}^2 \rightarrow \mathbb{Q}\Lambda(b)/\Lambda(b), \quad \phi((x, y)) = ux + vy - by.$$ 

When $b \notin \mathbb{Q}$, the subspace $\mathbb{Q}\Lambda(b) \subset \mathbb{R}$ is two dimensional over $\mathbb{Q}$ and one defines a character on the $\mathbb{Q}$-rational points by setting $\chi(x - by) := y \in \mathbb{Q}/\mathbb{Z}$, for $x - by \in \mathbb{Q}\Lambda(b)/\Lambda(b)$. Using this character one gets $\chi \circ \phi((x, y)) = y$ and this condition characterizes the relevant pseudolattices.
Lift of Frobenius correspondences

The Frobenius correspondences make sense in characteristic 1. The right action of the diagonal part of $P^+(\mathbb{R})$ deforms the complex structure of the foliation and suggests to analyze this deformation as dequantization. This situation is familiar in tropical geometry and at the algebraic level it suggests to define the structure sheaf of the proetale cover of the punctured disk using the Puiseux series as a coefficient field.
Frobenius correspondences

Use the Witt construction in characteristic 1, entropy

\[ u + v = \sup_{\alpha \in [0,1]} c(\alpha)u^\alpha v^{1-\alpha}, \quad c(\alpha) := \alpha^{-\alpha}(1-\alpha)^{1-\alpha} \]

Automorphisms \( \theta_{\lambda} \in \text{Aut}(W) \), Teichmüller lift \([x]\)

\[ \theta_{\lambda}([x]) = [x^\lambda], \quad \forall x \in \mathbb{R}_+^{\max}, \lambda \in \mathbb{R}_+. \]

The right action \( R(\mu) \) of \( \mathbb{R}_+^* \subset P_+(\mathbb{R}) \) extends to \( W \) valued functions, the arithmetic Frobenius is

\[ f \mapsto \text{Fr}_{\mu}^a(f), \quad \text{Fr}_{\mu}^a(f) := \theta_{\mu}(R(\mu^{-1})(f)) \]

\[ q(x + iy) := [e^{-2\pi y}]e^{2\pi ix} \]
1. Develop intersection theory in such a way that the divergent term in log Λ is eliminated.

2. Formulate and prove a Hirzebruch-Riemann-Roch formula on the square whose topological side part $\frac{1}{2}c_1(E)^2$ is $\frac{1}{2}s(f, f)$. This step involves the lifting $D(f) = \int f(\lambda)\Psi_\lambda d^*\lambda$ to a divisor $\tilde{D}(f)$ in the complex set-up and the use of correspondences.

3. Use the assumed positivity of $s(f, f)$ to get an existence result for $H^0(\tilde{D}(f))$ or $H^0(-\tilde{D}(f))$.

4. Use tropical descent to get the effectivity of a divisor equivalent to $D(f)$ and finally get a contradiction.