Cohomology to logic and back

Toposes in Como 2018

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Spaces have

- Euler characteristic
- Betti numbers
- homology groups
- cohomology rings(!)
- set of connected components
- fundamental group
- homotopy groups

Which of these generalize to toposes?

Why should they even generalize?

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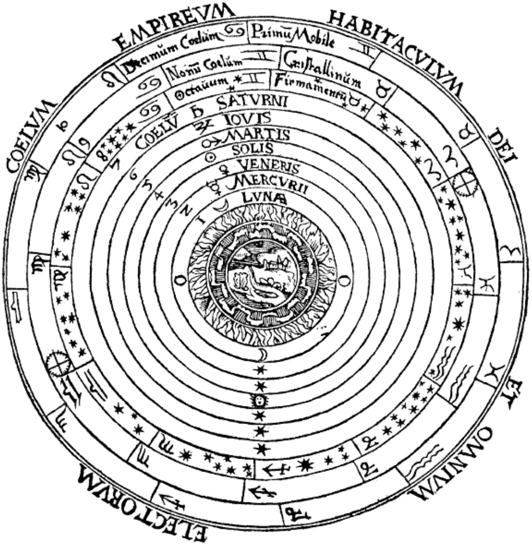

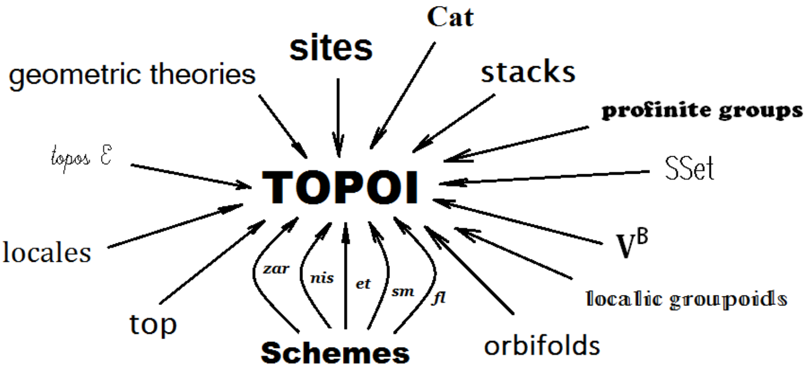
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- the category of Grothendieck toposes and geometric morphisms is excellent for representing categories of 'locally structured objects'
- functors $\mathcal{C} \to \mathrm{T{\small OPOI}}$ are seldom full ($\mathrm{T{\small OPOI}}$ is not locally small!)
- some of these functors are onto (up to isomorphism) on objects, allowing one to express $\mathrm{T{\small OPOI}}$ as a localization of $\mathcal{C}$
- $\mathrm{T{\small OPOI}}$ is a 2-category: given morphisms $f, g : \mathcal{E} \to \mathcal{F}$, a 2-arrow is a natural transformation from f^* to g^*.

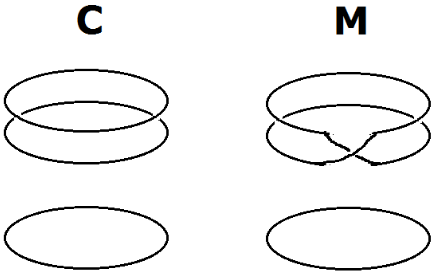
- the category of Grothendieck toposes and geometric morphisms is excellent for representing categories of 'locally structured objects'
- functors $\mathcal{C} \to \textsc{Topoi}$ are seldom full ($\textsc{Topoi}$ is not locally small!)
- some of these functors are onto (up to isomorphism) on objects, allowing one to express $\textsc{Topoi}$ as a localization of $\mathcal{C}$
- $\textsc{Topoi}$ is a 2-category: given morphisms $f, g : \mathcal{E} \to \mathcal{F}$, a 2-arrow is a natural transformation from f^* to g^*.

Challenge: can your favorite functor $\mathcal{C} \to \textsc{Topoi}$ be extended to a 2-functor?

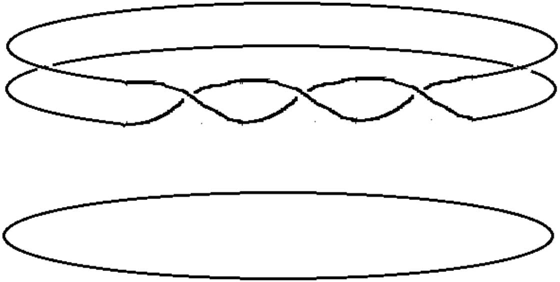
For example, can the category of topological spaces (of schemes, etc) be turned into a 2-category compatibly with the functor $X \mapsto \mathrm{Sh}(X)$ $\quad$ ($X \mapsto \mathrm{Sh}(X_{\acute{e}t})$ etc)?

$C := \{0,1\} \times [0,2\pi]$ modulo $\langle 0,0\rangle \sim \langle 0,2\pi\rangle$ and $\langle 1,0\rangle \sim \langle 1,2\pi\rangle$
$M := \{0,1\} \times [0,2\pi]$ modulo $\langle 0,0\rangle \sim \langle 1,2\pi\rangle$ and $\langle 1,0\rangle \sim \langle 0,2\pi\rangle$

natural projection to $S^1 = [0,2\pi]$ modulo $0 \sim 1$



C is not isomorphic to M in $\mathrm{Top}\,/S^1$.



Boundary of n times twisted Moebius strip is isomorphic in Top/S^1 to C for n even, and to M for n odd.

What are these objects? Why only two isomorphism types? What about bases other than S^1?

default assumptions All toposes are Grothendieck.
Base topos Set is model of ZFC. Additional universes as needed.

In topos $\mathcal{E}$, objects X and Y are *locally isomorphic* if there exists isomorphism $X \times U \approx Y \times U$ over U, for some $U \twoheadrightarrow \mathbf{1}$.

Canonical geometric morphism $\gamma : \mathsf{Sh}(S^1) \to \mathsf{Set}$. What the pictures showed (identifying an object of $\acute{E}t/S^1$ with its sheaf of continuous sections) were isomorphism classes of objects of $\mathsf{Sh}(S^1)$ that are locally isomorphic to $\gamma^*(\{0,1\})$.

Given a topos $\mathcal{E}$, let $\mathsf{Tors}_{\mathbb{Z}/2}(\mathcal{E})$ denote the subcategory of $\mathcal{E}$ whose objects are locally isomorphic to the two-element constant set, and whose morphisms are the isomorphisms between them. Let $\mathrm{H}^1(\mathcal{E}, \mathbb{Z}/2)$ denote the set(!) of isomorphism types of objects of $\mathsf{Tors}_{\mathbb{Z}/2}(\mathcal{E})$.

presheaf topos Pre($\mathcal{C}$)

- $T : \mathcal{C}^{\text{op}} \to \text{Set}$ is locally isomorphic to $\{0, 1\}$ iff $T(c) = \{0, 1\}$ for all $c \in \mathcal{C}$, and $T(f) \in \text{aut}(\{0, 1\})$ for all morphisms $c \xrightarrow{f} d$
- this amounts to functor $\mathcal{C}^{\text{op}} \to \mathbb{Z}/2$ (where a group is thought of as one-object category)
- isomorphism between two such objects amounts to natural isomorphism between the associated functors
- equivalence of categories $\text{Fun}(\mathcal{C}, \mathbb{Z}/2) \to \text{Tors}_{\mathbb{Z}/2}(\text{Pre}(\mathcal{C}))$

presheaf topos $\mathsf{Pre}(\mathcal{C})$

- equivalence of categories $\mathsf{Fun}(\mathcal{C}, \mathbb{Z}/2) \to \mathsf{Tors}_{\mathbb{Z}/2}(\mathsf{Pre}(\mathcal{C}))$
- Grpd is a reflective subcategory of Cat:

  $$\pi_1 : \mathsf{Cat} \leftrightarrows \mathsf{Grpd} : i$$

  $\pi_1(\mathcal{C})$ is the *fundamental groupoid* of the category $\mathcal{C}$, obtained by formally adjoining an inverse to each morphism of $\mathcal{C}$.
- equivalence of categories $\mathsf{Grpd}(\pi_1(\mathcal{C}), \mathbb{Z}/2) \to \mathsf{Tors}_{\mathbb{Z}/2}(\mathsf{Pre}(\mathcal{C}))$
- bijection between isomorphism types of objects (i.e. conjugacy classes of homomorphisms) in $\mathsf{Grpd}(\pi_1(\mathcal{C}), \mathbb{Z}/2)$ and $\mathrm{H}^1(\mathsf{Pre}(\mathcal{C}), \mathbb{Z}/2)$.

$$\frac{\mathrm{H}^1(\mathcal{E}, G)}{\text{conj classes in } \mathsf{Grpd}(\pi_1(\mathcal{C}), G)}$$

spatial topos Sh(X)

What is the set of $\mathbb{Z}/2$-torsors T that are trivialized by a given open cover $\{U_i \mid i \in I\}$ of X?

By assumption, T restricted to U_i is represented by the étale space $U_i \times \{0, 1\}$. ('0' and '1' are arbitrary labels, not assumed globally defined!) When do these étale spaces 'glue' together to a two-sheeted cover?

To every non-empty intersection $U_i \cap U_j$, associate $g_{ij} \in \mathrm{aut}(\{0, 1\})$. If the 'cocycle conditions'

- $g_{ii} = \mathrm{id}$ and $g_{ij}^{-1} = g_{ji}$
- $g_{ij} g_{jk} = g_{ik}$ for non-empty $U_i \cap U_j \cap U_k$

are satisfied then the g_{ij} define an equivalence relation on $\sqcup_{i \in I} U_i \times \{0, 1\}$, and quotient is an étale space over X with two-point fibers.

spatial topos $\mathrm{Sh}(X)$

- two such cocycles g_{ij}, g'_{ij} result in spaces isomorphic over X iff there exist $h_i \in \mathrm{aut}(\{0,1\})$ such that $h_i g_{ij} = g'_{ij} h_j$
- refinement of cover $\mathcal{U} = \{U_i \,|\, i \in I\}$ by $\mathcal{V} = \{V_j \,|\, j \in J\}$, given by $J \to I$, induces map from the set of $\mathbb{Z}/2$-torsors trivialized over $\mathcal{U}$ to the set of $\mathbb{Z}/2$-torsors trivialized over $\mathcal{V}$
- given $\mathcal{U}$ and $\mathcal{V}$, induced map does not depend on $J \to I$
- since any torsor must be trivialized over *some* open cover,

$$\mathrm{H}^1(\mathrm{Sh}(X), \mathbb{Z}/2) = \mathrm{colim}_{\mathcal{U}}\, \check{\mathrm{Z}}(\mathcal{U}, \mathbb{Z}/2)/\check{\mathrm{B}}(\mathcal{U}, \mathbb{Z}/2) =: \check{\mathrm{H}}^1(\mathrm{Sh}(X), \mathbb{Z}/2)$$

$\mathrm{H}^1(\mathcal{E}, G)$

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conj classes in $\mathrm{Grpd}(\pi_1(\mathcal{C}), G)$

$\check{\mathrm{H}}^1(X, G)$

simplicial coding of torsors

Let $[n] = \{0 < 1 < 2 < \cdots < n\}$ and Δ the category with objects $[n]$, $n \in \mathbb{N}$, and morphisms monotone maps.

SSet is shorthand for $\mathrm{Set}^{\Delta^{\mathrm{op}}}$.

- Δ_n is the simplicial complex consisting of all (non-empty) subsets of $\{0, 1, \ldots, n\}$
- $\Lambda_{n,k}$ is Δ_n omitting $\{0, 1, \ldots, n\}$ and $\{0, 1, \ldots, k-1, k+1, \ldots, n\}$
- these determine simplicial sets, denoted the same way.

simplicial coding of torsors

$X \xrightarrow{f} Y \in \mathrm{SSet}$ is a *Kan fibration* if it satisfies the lifting property

$$\begin{array}{ccc} \Lambda_{n,k} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta_n & \longrightarrow & Y \end{array}$$

for all $n \in \mathbb{N}$, $0 \leqslant k \leqslant n$.

It is an *N-exact Kan fibration* if, in addition, the dotted lift is *unique* for all commutative squares with $n > N$.

This is equivalent to the canonical map $X(n) \to \Lambda_{n,k}(f)$ being epi for all n, and iso for $n > N$. This is the definition we will use for simplicial objects in a topos.

simplicial coding of torsors

- for group object G in a topos $\mathcal{E}$, write $K(G,1)$ for its nerve, considered as a one-object category
- for object U of $\mathcal{E}$, $U_\bullet = \mathrm{cosk}_0(U)$ is the standard simplicial object $\cdots U^3 \Rrightarrow U^2 \rightrightarrows U$

Theorem [Duskin 1975] There is a natural bijection between isomorphism classes G-torsors trivialized over U, and simplicial homotopy classes of 1-exact fibrations of the form $U_\bullet \to K(G,1)$.

$$\mathrm{H}^1(\mathcal{E}, G) = \mathrm{colim}_{[U_\bullet]} \, [-, K(G,1)]$$

i.e. the filtered colimit, along homotopy classes of simplicial covers, of simplicial homotopy classes of maps into $K(G,1) \in \mathcal{E}^{\Delta^{\mathrm{op}}}$.

$\mathrm{H}^1(\mathcal{E}, G)$

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conj classes in $\mathrm{Grpd}(\pi_1(\mathcal{C}), G)$

$\check{\mathrm{H}}^1(X, G)$

$\mathrm{colim}_{[U_\bullet]} \; [-, K(G,1)]$

relation to Quillen model categories

$X \xrightarrow{f} Y \in \operatorname{mor} \mathcal{E}^{\Delta^{\mathrm{op}}}$ is a *local* or *internal* weak homotopy equivalence if f induces isomorphisms on the homotopy group objects (constructed internally) with respect to arbitrary (local) basepoints.

Let $\mathrm{W} \subset \operatorname{mor} \mathcal{E}^{\Delta^{\mathrm{op}}}$ denote the class of local weak equivalences. W is the class of models in $\operatorname{mor} \mathcal{E}^{\Delta^{\mathrm{op}}}$ of a countable geometric sketch. See e.g. [TB 2000] for an axiomatization by sentences of geometric logic.

Theorem [Joyal ca. 1984] Setting cofibrations to be the monomorphisms and weak equivalences to be W, $\mathcal{E}^{\Delta^{\mathrm{op}}}$ satisfies Quillen's axioms for a *homotopy model category*.

relation to Quillen model categories

Corollary $\mathrm{ho}_{\mathcal{E}^{\Delta^{op}}} := \mathcal{E}^{\Delta^{op}}[W^{-1}]$ is a locally small category.

$\mathcal{E}^{\Delta^{op}}[W^{-1}]$ denotes the category $\mathcal{E}^{\Delta^{op}}$ where morphisms belonging to W have been formally inverted. It is the non-additive analogue of the derived category of ($\mathbb{N}$-graded) chain complexes of abelian group objects in $\mathcal{E}$.

Theorem $\mathrm{H}^1(\mathcal{E}, G) = \mathrm{ho}_{\mathcal{E}^{\Delta^{op}}}\big(\mathbf{1}, K(G,1)\big)$

See e.g. [Jardine 1986] for a proof using homological algebra. There is a proof [TB 2001] by simplicial manipulations.

$$\frac{\mathrm{H}^1(\mathcal{E}, G)}{}$$

conj classes in $\mathsf{Grpd}(\pi_1(\mathcal{C}), G)$

$\check{\mathrm{H}}^1(X, G)$

$\mathrm{colim}_{[U_\bullet]} \left[-, K(G,1)\right]$

$\mathrm{ho}_{\mathcal{E}^{\Delta^{op}}} \left(\mathbf{1}, K(G,1)\right)$

classical definition

Let $(\mathcal{C}, J)$ be a site and G a sheaf of groups on $\mathcal{C}$. A *pseudo G-torsor* is a sheaf of sets T on $\mathcal{C}$ endowed with an action $G \times T \to T$ such that whenever $T(U)$ is nonempty then the action $G(U) \times F(U) \to F(U)$ is simply transitive. A *G-torsor* is a pseudo G-torsor such that for every $U \in \mathrm{ob}(\mathcal{C})$ there exists a J-covering $\{U_i \to U\}_{i \in I}$ of U such that $T(U_i)$ is nonempty for all $i \in I$. A morphism of (pseudo) G-torsors $T \to T'$ is a morphism of sheaves of sets compatible with the G-actions.

See e.g. [de Jong]. In applications, often $\mathcal{C}$ is Schemes, J is the Zariski, étale, fppf, fpqc etc. topology, G is represented by a group scheme and T by a scheme.

concise definition

Let G be any group object in the topos $\mathcal{E}$. A G-torsor is an object T with a G-action $G \times T \xrightarrow{\mu} T$ satisfying

$$T \twoheadrightarrow \mathbf{1}$$

$$G \times T \xrightarrow{\mu \times \mathrm{pr}_2} T \times T \text{ is an isomorphism.}$$

This implies that T is 'locally trivial', i.e. isomorphic to G acting on itself (by base change to T itself).

equivalently, a G-torsor is . . .

- an $\mathcal{E}$-valued flat presheaf on the (internal) category G
- a model in $\mathcal{E}$ of the single-sorted (internal) geometric theory T_G, with $g \in G$ as unary function symbols
  - $\exists x \in T$
  - $\forall x\, e(x) = x$ where e is the identity element in G
  - $\forall x\, g(h(x)) = (gh)(x)$
  - $\forall x \forall y \big(\bigvee_{g \in G} g(x) = y \big)$
  - $\forall x \big(g(x) = h(x) \implies \bot \big)$ for each $g \neq h \in G$.

note on the theory T_G

(1) any homomorphism of T_G-models, in any topos $\mathcal{E}$, is an isomorphism: $\mathrm{Mod}_{T_G}(\mathcal{E})$ is a groupoid

(2) in any topos $\mathcal{E}$, there is only a *set* of isomorphism types of T_G-models.

This is unusual, even in Set! By Löwenheim-Skolem, if a first order theory has a model of cardinality $\geqslant \max\{\aleph_0, \text{size of the signature}\}$ then it has a proper class of non-isomorphic models. (Analogues for infinitary logics via Hanf number.)

Can one characterize the geometric theories with the property that any homomorphism of models is an isomorphism?

(1) implies (2)

Proposition Let $\mathcal{E}$ be a locally presentable category and $\mathcal{S}$ a sketch. If $\mathrm{Mod}_{\mathcal{S}}(\mathcal{E})$ is a groupoid, then it is a *small* groupoid. (That is, it only has a set of isomorphism classes of objects.)

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Proof $\mathrm{Mod}_{\mathcal{S}}(\mathcal{E})$ is an accessible category [Makkai-Paré 1989]. Every accessible category has a small dense subcategory. If $\mathrm{Mod}_{\mathcal{S}}(\mathcal{E})$ is a groupoid with D as dense set of generators then

$$\mathrm{card}\left(\text{iso-classes in } \mathrm{Mod}_{\mathcal{S}}(\mathcal{E})\right) \leqslant \mathrm{card}(D)\ .$$

Write $\mathrm{GeoMor}(\mathcal{E}, \mathcal{F})$ for the category of geometric morphisms from $\mathcal{E}$ to $\mathcal{F}$ and natural transformations.

Corollary Let $\mathcal{E}$, $\mathcal{F}$ be toposes. If $\mathrm{GeoMor}(\mathcal{E}, \mathcal{F})$ is a groupoid, then it is an (essentially) small groupoid.

torsors via classifying toposes

The classifying topos of the theory of G-torsors (for constant G) is G-Set.

There's an equivalence of categories (groupoids!)

$$\mathsf{Tors}_G(\mathcal{E}) = \mathsf{GeoMor}(\mathcal{E}, \text{G-Set})$$

and a canonical bijection

$$\mathrm{H}^1(\mathcal{E}, G) = \text{isomorphism classes of objects in } \mathsf{GeoMor}(\mathcal{E}, \text{G-Set}) .$$

"neo-classical" definition

Torsors are to groups as affine spaces are to vector spaces: a set with operations that becomes a group as soon as an element is chosen. It is possible to define *torsor* without making the structure group G explicit:

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Torsors are to groups as affine spaces are to vector spaces: a set with operations that becomes a group as soon as an element is chosen. It is possible to define *torsor* without making the structure group G explicit:

Definition [Baer 1929] A *torsor* is a non-empty set T equipped with a ternary operation $T^3 \to T$, denoted $\langle -, -, - \rangle$, satisfying

$$\langle x, y, y \rangle = \langle y, y, x \rangle = x$$

$$\langle \langle x, y, z \rangle, u, v \rangle = \langle x, \langle u, z, y \rangle, v \rangle = \langle x, y, \langle z, u, v \rangle \rangle$$

“neo-classical” definition

Any group G becomes a Baer torsor under $\langle g, h, k\rangle := gh^{-1}k$.

There’s a complicated but explicit (choice-free!) recipe for constructing a group G and set X from a Baer torsor T so that X becomes a G-torsor. See e.g. [Huber–Müller-Stach 2017].

exercise

- Determine the classifying topos $\mathbb{B}[B]$ of Baer torsors.
- Given a group G in Set, describe the geometric morphism G-Set $\to \mathbb{B}[B]$ classifying the generic G-torsor as a Baer torsor.
- Is the theory of Baer torsors Morita-equivalent to the two-sorted theory of torsors? In more detail: the sorts are G and X, one constant $e \in G$, unary $()^{-1} : G \to G$, binary operations $G \times G \to G$ and $G \times X \to X$ satisfying that X is non-empty and
  - $eg = ge = g$
  - $gg^{-1} = g^{-1}g = e$
  - $g_1(g_2g_3) = (g_1g_2)g_3$
  - $e(x) = x$
  - $g_1(g_2(x)) = (g_1g_2)(x)$
  - $\forall x, y \in X\, \exists! g \in G \big(g(x) = y\big)$

$\mathrm{H}^1(\mathcal{E}, G)$	
conj classes in $\mathrm{Grpd}(\pi_1(\mathcal{C}), G)$	
$\check{\mathrm{H}}^1(X, G)$	
$\operatorname{colim}_{[U_\bullet]} \left[-, K(G,1)\right]$	
$\mathrm{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}\left(\mathbf{1}, K(G,1)\right)$	
iso classes in $\mathrm{GeoMor}(\mathcal{E}, \text{G-Set})$	

$\mathrm{H}^1(\mathcal{E}, G)$	$\mathrm{H}^n(\mathcal{E}, G),\ n > 1$
conj classes in $\mathrm{Grpd}(\pi_1(\mathcal{C}), G)$	
$\check{\mathrm{H}}^1(X, G)$	
$\mathrm{colim}_{[U_\bullet]} \left[-, K(G, 1)\right]$	
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$\mathrm{H}^1(\mathcal{E}, G)$		$\mathrm{H}^n(\mathcal{E}, G),\ n > 1$
conj classes in $\mathrm{Grpd}(\pi_1(\mathcal{C}), G)$	[!]	
$\check{\mathrm{H}}^1(X, G)$		
$\mathrm{colim}_{[U_\bullet]} \left[-, K(G,1)\right]$		
$\mathrm{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}} \left(\mathbf{1}, K(G,1)\right)$		
iso classes in $\mathrm{GeoMor}(\mathcal{E}, \text{G-Set})$		

some theories of the fundamental group of a topos

- profinite fundamental group [SGA1; Johnstone 1977]
- "Revêtements principaux galoisiens infinis et groupe fondamental élargi" [SGA3.X.6]
- fundamental pro-group [Artin-Mazur 1969]
- path fundamental group [Moerdijk-Wraith 1986]
- prodiscrete fundamental group [Moerdijk 1989]
- localic fundamental groupoid [Bunge 1992]
- localic fundamental groupoid [Kennison 1992]
- fundamental pro-groupoid [Chirvasitu-Johnson-Freyd 2012]

$\pi_1(X, x)$ for topological space X: no obvious universal property

Seifert-van Kampen theorem (in classical form) states that if topological space $X = A \cup B$ with A, B open, A, B and $A \cap B$ path connected, $x \in A \cap B$, then

$$\pi_1(X, x) = \pi_1(A, x) *_{\pi_1(A \cap B, x)} \pi_1(B, x)$$

(as pushout, or "amalgamated sum"). Proof needs compactness of unit interval!

For simplicial sets, adjunction

$$\pi_1 : \mathrm{SSet} \leftrightarrows \mathrm{Grpd} : i$$

For topological spaces, $\pi_1(-, *)$ commutes with arbitrary products. For simplicial sets, it does not. For Kan (i.e. fibrant), simplicial sets, it does. What is "right"?

wishlist for the notion of fundamental group

- should be basepoint-free (i.e. fundamental *groupoid*)
- $\mathrm{H}^1(\mathcal{E}, G)$ should biject with (isomorphism classes of) morphisms from $\pi_1(\mathcal{E})$ to G
- for spatial, locally simply connected toposes, ought to be groupoid in Set
- Seifert-van Kampen should hold in some form
- should generalize to the category of all [?] toposes over a base topos

problem

- Identify the class $\textsc{Galois}$ of toposes $\mathcal{F}$ with the property that for every topos $\mathcal{E}$, the category $\mathrm{GeoMor}(\mathcal{E}, \mathcal{F})$ is a groupoid. (If connected, these toposes could be thought of as the analogues of Eilenberg-MacLane $K(G, 1)$ spaces.)
- Construct a (2-)reflection

$$\pi_1 : \textsc{Topoi} \leftrightarrows \textsc{Galois} : i$$

  The unit $\mathcal{E} \to \pi_1(\mathcal{E})$ can be viewed as classifying the "universal torsor", as a Postnikov 1-section of $\mathcal{E}$.
- Of the existing theories of the fundamental group(oid), which ones satisfy this universal mapping property (possibly when restricted to connected, locally connected toposes)?

natural homotopy

Definition (Joyal) Let $\mathcal{E}$, $\mathcal{F}$ be toposes. Two geometric morphisms $f, g : \mathcal{E} \to \mathcal{F}$ are *naturally homotopic* if there exists a natural transformation between f^* and g^* (equivalently, between f_* and g_*).

$$[\mathcal{E}, \mathcal{F}] := \pi_0 \operatorname{GeoMor}(\mathcal{E}, \mathcal{F})$$

denotes the set(!) of connected components of $\operatorname{GeoMor}(\mathcal{E}, \mathcal{F})$, i.e. the class of geometric morphisms from $\mathcal{E}$ to $\mathcal{F}$, modulo the equivalence relation generated by natural homotopy.

$$[\mathcal{E}, \mathcal{F}] := \pi_0 \, \mathsf{GeoMor}(\mathcal{E}, \mathcal{F})$$

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- Fix $\mathcal{F}$ and think of $\mathcal{E}$ as variable. $[-, \mathcal{F}]$ is a set-valued invariant of toposes. Several classical (contravariant) invariants are so representable!
- $\mathcal{F}$ may be part of some structured data (e.g. suitable diagram of toposes), inducing algebraic structure on the sets $[-, \mathcal{F}]$.
- Conversely, think of $\mathcal{F} = \mathbb{B}[T]$ as a classifying topos and $\mathcal{E}$ as a model of 'set theory'. $\mathrm{Mod}_T(\mathcal{E})$ may be an interesting category, with many connected components, even if $\mathrm{Mod}_T(\mathrm{Set})$ is connected.
- Sometimes (rare!) $\mathrm{Mod}_T(\mathcal{E})$ is a groupoid, and $\big[\mathcal{E}, \mathbb{B}[T]\big]$ bijects with isomorphism types of T-models in $\mathcal{E}$.

exercise

$\mathcal{E}$ and $\mathcal{F}$ are *naturally homotopy equivalent* if there exist geometric morphisms $\mathcal{E} \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{E}$ whose composites are naturally homotopic to the respective identity.

Let T be a finite limit theory. Show that its classifying topos $\mathbb{B}[T]$ is naturally homotopy equivalent to Set (the "point").

It follows that all "algebraic topological" invariants of T derived from its classifying topos (cohomology groups, fundamental groupoid) are trivial.

Proposition (Joyal) Fix $n \in \mathbb{N}$ and $A \in \mathsf{Ab}$. There exist toposes $\mathcal{K}(A, n)$ such that there is a natural bijection

$$\mathrm{H}^n(\mathcal{E}, A) = \left[\, \mathcal{E}, \mathcal{K}(A, n) \,\right]$$

That is, (abelian) cohomology (with constant coefficients) is representable via natural homotopy classes.

Any two *Eilenberg–MacLane toposes* $\mathcal{K}(A, n)$, i.e. toposes with the above property, are naturally homotopy equivalent.

example of an Eilenberg–MacLane topos

A *Yoneda extension of length n of* $\mathbb{Z}$ *by A* is a diagram of abelian group objects

$$0 \to A \to E_0 \to E_1 \to \cdots \to E_{n-1} \to \mathbb{Z} \to 0$$

that is exact at every object. A morphism of Yoneda extensions is a commutative 'ladder' from one such short exact sequence to another.

$\mathrm{H}^n(\mathcal{E}, A)$ is $Ext^n_{\mathrm{Ab}(\mathcal{E})}(\mathbb{Z}, A)$, which bijects with zig-zag equivalence classes of Yoneda extensions, which bijects with the class of Yoneda extensions modulo the equivalence relation generated by morphisms of Yoneda extensions.

Let $\mathbb{B}[Y(A, n)]$ be the classifying topos of the theory of Yoneda extensions of length n of $\mathbb{Z}$ by A. The above facts mean *exactly* that $\mathbb{B}[Y(A, n)]$ is an Eilenberg–MacLane topos $\mathcal{K}(A, n)$!

another example of an Eilenberg–MacLane topos

One can also take $\mathcal{K}(A, n)$ to be the classifying topos of the theory of simplicial maps of the form $C \to K(A, n)$ where $K(A, n)$ is the standard simplicial model and C is (internally) weakly contractible. [Joyal-Wraith 1984]

$K(A, n)$ can be replaced by any internally fibrant simplicial object X, to result in [TB 2004]

$$\left[\mathcal{E}, \mathbb{B}[\mathrm{ac}/X]\right] = \mathrm{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$$

Theorem [Joyal-Wraith 1984]

Let $\mathcal{K}(A, n)$ be any Eilenberg-MacLane topos, and let $K(A, n)$ be any Eilenberg-MacLane space. The geometric morphism (unique up to natural homotopy)

$$\mathrm{Sh}(K(A, n)) \to \mathcal{K}(A, n)$$

classifying the tautologous class in $\mathrm{H}^n(K(A, n), A)$, induces a weak homotopy equivalence on pro-homotopy types in the sense of Artin and Mazur.

It follows that the cohomology rings of Eilenberg-MacLane toposes and Eilenberg-MacLane spaces are isomorphic.

$\mathrm{H}^1(\mathcal{E}, G)$	$\mathrm{H}^n(\mathcal{E}, A)$, $n > 1$
conj classes in $\mathrm{Grpd}(\pi_1(\mathcal{E}), G)$	(Hurewicz theorem is "analogous")
$\check{\mathrm{H}}^1(\mathcal{E}, G)$	Čech cohomology computed along n-truncated hypercovers
$\mathrm{colim}_{[U_\bullet]} \left[-, K(G,1)\right]$	$\mathrm{colim}_{[\mathrm{cov}_n(\mathcal{E})]} \left[-, K(A,n)\right]$
$\mathrm{ho}_{\mathcal{E}^{\Delta^{op}}} \left(\mathbf{1}, K(G,1)\right)$	$\mathrm{ho}_{\mathcal{E}^{\Delta^{op}}} \left(\mathbf{1}, K(A,n)\right)$
iso classes in $\mathrm{GeoMor}(\mathcal{E}, \text{G-Set})$	$\pi_0 \, \mathrm{GeoMor}\left(\mathcal{E}, \mathcal{K}(A,n)\right)$

conjecture

For no $n > 1$ and (non-zero) abelian group A does there exist a topos $\mathcal{K}$ with a natural bijection

$$\mathrm{H}^n(\mathcal{E}, A) = \text{ isomorphism classes in } \mathrm{GeoMor}(\mathcal{E}, \mathcal{K}) \; .$$

That is, only with H^1 does one have the good luck of 'counting' isomorphism types of 'algebraic gadgets'.

With H^n $(n > 1)$, one necessarily counts equivalence classes of 'algebraic gadgets' under an algebraically definable relation (satisfying a calculus of fractions, perhaps).

The intuition is that $\mathrm{GeoMor}(-, \mathcal{F})$ is a groupoid *only* when $\mathcal{F}$ classifies torsors.

heuristic *(rough!)*

- topos $\mathcal{F}$ is such that for all toposes $\mathcal{E}$, the category $\mathrm{GeoMor}(\mathcal{E}, \mathcal{F})$ has one isomorphism type of object in each connected component
- topos $\mathcal{F}$ is such that for all toposes $\mathcal{E}$, the category $\mathrm{GeoMor}(\mathcal{E}, \mathcal{F})$ is a groupoid
- $\mathcal{F}$ is a Galois topos (suitably interpreted!)
- $\mathcal{F}$ has the natural homotopy type of a $\mathcal{K}(\mathcal{G}, 1)$
- $\mathcal{F}$ cannot be an Eilenberg-MacLane topos $\mathcal{K}(A, n)$ for $n > 1$.

If each line could be made precise and imply the next, one would be done!

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