Infinitary generalizations of Deligne's completeness theorem

Christian Espíndola

Postdoctoral researcher at Masaryk University

June 29th, 2018

Deligne's completeness theorem

Deligne's completeness theorem

A coherent topos is a topos of sheaves for which there is a site on a category with finite limits and where the Grothendieck topology has a basis consisting of finite covering families.

Deligne's completeness theorem

A coherent topos is a topos of sheaves for which there is a site on a category with finite limits and where the Grothendieck topology has a basis consisting of finite covering families.
A point of a topos $\mathcal{E}$ is a geometric morphism $\mathcal{S}et \to \mathcal{E}$. The topos has enough points if the collection of points is jointly surjective, i.e., the inverse images of the points are jointly conservative.

Deligne's completeness theorem

A coherent topos is a topos of sheaves for which there is a site on a category with finite limits and where the Grothendieck topology has a basis consisting of finite covering families.
A point of a topos $\mathcal{E}$ is a geometric morphism $\mathcal{S}et \to \mathcal{E}$. The topos has enough points if the collection of points is jointly surjective, i.e., the inverse images of the points are jointly conservative. This means that for two subobjects S, T of a given object, we have:

$$S \leq T \iff p^*(S) \leq p^*(T) \text{ for every point } p$$

Deligne's completeness theorem

A coherent topos is a topos of sheaves for which there is a site on a category with finite limits and where the Grothendieck topology has a basis consisting of finite covering families.
A point of a topos $\mathcal{E}$ is a geometric morphism $\mathcal{S}et \to \mathcal{E}$. The topos has enough points if the collection of points is jointly surjective, i.e., the inverse images of the points are jointly conservative. This means that for two subobjects S, T of a given object, we have:

$$S \leq T \iff p^*(S) \leq p^*(T) \text{ for every point } p$$

Theorem

(Deligne) A coherent topos has enough points.

Deligne's completeness theorem

A coherent topos is a topos of sheaves for which there is a site on a category with finite limits and where the Grothendieck topology has a basis consisting of finite covering families.

A point of a topos $\mathcal{E}$ is a geometric morphism $\mathcal{S}et \to \mathcal{E}$. The topos has enough points if the collection of points is jointly surjective, i.e., the inverse images of the points are jointly conservative. This means that for two subobjects S, T of a given object, we have:

$$S \leq T \iff p^*(S) \leq p^*(T) \text{ for every point } p$$

Theorem

(Deligne) A coherent topos has enough points.

Essentially equivalent to Gödel completeness theorem for first-order classical logic.

Generalizations

A separable topos is a topos of sheaves for which there is a site on a category with countably many objects and morphisms, and where the Grothendieck topology is generated by countably many covering families.

A separable topos is a topos of sheaves for which there is a site on a category with countably many objects and morphisms, and where the Grothendieck topology is generated by countably many covering families.

Theorem

(Makkai-Reyes) A separable topos has enough points.

Generalizations

A separable topos is a topos of sheaves for which there is a site on a category with countably many objects and morphisms, and where the Grothendieck topology is generated by countably many covering families.

Theorem

(Makkai-Reyes) A separable topos has enough points.

Essentially equivalent to Karp completeness theorem for infinitary classical logic over $\mathcal{L}_{\omega_1,\omega}$. It also entails completeness of countably axiomatized geometric theories.

Generalizations

A separable topos is a topos of sheaves for which there is a site on a category with countably many objects and morphisms, and where the Grothendieck topology is generated by countably many covering families.

Theorem

(Makkai-Reyes) A separable topos has enough points.

Essentially equivalent to Karp completeness theorem for infinitary classical logic over $\mathcal{L}_{\omega_1,\omega}$. It also entails completeness of countably axiomatized geometric theories.

GOAL: Replace ω by any cardinal κ with $\kappa^{<\kappa} = \kappa$, obtaining thereby a completeness theorem for infinitary classical logic over $\mathcal{L}_{\kappa^+,\kappa}$, and moreover, for an intuitionistic fragment that we call κ-geometric.

κ-geometric logic

κ-geometric logic

Extension of geometric logic in which we have:

κ-geometric logic

Extension of geometric logic in which we have:

- Arities of cardinality less than κ

κ-geometric logic

Extension of geometric logic in which we have:

- Arities of cardinality less than κ
- Conjunction of less than κ many formulas

κ-geometric logic

Extension of geometric logic in which we have:

- Arities of cardinality less than κ
- Conjunction of less than κ many formulas
- Existential quantification of less than κ many variables

κ-geometric logic

Extension of geometric logic in which we have:

- Arities of cardinality less than κ
- Conjunction of less than κ many formulas
- Existential quantification of less than κ many variables
- More logical axioms or rules are needed.

κ-geometric logic

Extension of geometric logic in which we have:

- Arities of cardinality less than κ
- Conjunction of less than κ many formulas
- Existential quantification of less than κ many variables
- More logical axioms or rules are needed.

Example: the theory of well-orderings:

κ-geometric logic

Extension of geometric logic in which we have:

- Arities of cardinality less than κ
- Conjunction of less than κ many formulas
- Existential quantification of less than κ many variables
- More logical axioms or rules are needed.

Example: the theory of well-orderings:

$$\top \vdash_{xy} x < y \vee y < x \vee x = y$$

$$\exists x_0 x_1 x_2 \ldots \bigwedge_{n \in \omega} x_{n+1} < x_n \vdash \bot$$

κ-geometric logic

Extension of geometric logic in which we have:

- Arities of cardinality less than κ
- Conjunction of less than κ many formulas
- Existential quantification of less than κ many variables
- More logical axioms or rules are needed.

Example: the theory of well-orderings:

$$\top \vdash_{xy} x < y \vee y < x \vee x = y$$

$$\exists x_0 x_1 x_2 \ldots \bigwedge_{n \in \omega} x_{n+1} < x_n \vdash \bot$$

López-Escobar: the theory of well-orderings is not axiomatizable in $\mathcal{L}_{\kappa,\omega}$ for any κ.

κ-geometric logic

κ-geometric logic

To summarize:

- More expressive power will require more axioms

κ-geometric logic

To summarize:

- More expressive power will require more axioms
- The new axioms in the κ-geometric fragment takes the form of a rule of inference

κ-geometric logic

To summarize:

- More expressive power will require more axioms
- The new axioms in the κ-geometric fragment takes the form of a rule of inference
- The rule will correspond to an exactness property of the associated category

κ-geometric logic

To summarize:

- More expressive power will require more axioms
- The new axioms in the κ-geometric fragment takes the form of a rule of inference
- The rule will correspond to an exactness property of the associated category
- This gives rise to κ-geometric categories

κ-geometric logic

To summarize:

- More expressive power will require more axioms
- The new axioms in the κ-geometric fragment takes the form of a rule of inference
- The rule will correspond to an exactness property of the associated category
- This gives rise to κ-geometric categories
- κ-geometric categories will have a κ-classifying topos

κ-geometric logic

To summarize:

- More expressive power will require more axioms
- The new axioms in the κ-geometric fragment takes the form of a rule of inference
- The rule will correspond to an exactness property of the associated category
- This gives rise to κ-geometric categories
- κ-geometric categories will have a κ-classifying topos

This leads us to the notion of κ-separable topos.

κ-geometric logic

To summarize:

- More expressive power will require more axioms
- The new axioms in the κ-geometric fragment takes the form of a rule of inference
- The rule will correspond to an exactness property of the associated category
- This gives rise to κ-geometric categories
- κ-geometric categories will have a κ-classifying topos

This leads us to the notion of κ-separable topos.

A κ-separable topos is a topos of sheaves for which there is a site on a category with κ-small limits, at most κ many objects and morphisms, where the basis for the Grothendieck topology is generated by at most κ-many covering families and satisfying a further exactness property T.

κ-geometric logic

To summarize:

- More expressive power will require more axioms
- The new axioms in the κ-geometric fragment takes the form of a rule of inference
- The rule will correspond to an exactness property of the associated category
- This gives rise to κ-geometric categories
- κ-geometric categories will have a κ-classifying topos

This leads us to the notion of κ-separable topos.

A κ-separable topos is a topos of sheaves for which there is a site on a category with κ-small limits, at most κ many objects and morphisms, where the basis for the Grothendieck topology is generated by at most κ-many covering families and satisfying a further exactness property T.

Theorem

(E.) κ-separable toposes have enough κ-points.

The exactness property T

The exactness property T

Given a locale L, we would like to define points that in addition preserve κ-small intersections.

The exactness property T

Given a locale L, we would like to define points that in addition preserve κ-small intersections. These correspond to κ-complete, completely prime filters of L.

The exactness property T

Given a locale L, we would like to define points that in addition preserve κ-small intersections. These correspond to κ-complete, completely prime filters of L.

This leads us to a strong distributivity property:

$$\frac{\begin{array}{ll} \phi_f \vdash_{\mathbf{x}} \bigvee_{g \in \gamma^{\beta+1}, g|_\beta = f} \phi_g & \beta < \gamma, f \in \gamma^\beta \\ \phi_f \dashv\vdash_{\mathbf{x}} \bigwedge_{\alpha < \beta} \phi_{f|_\alpha} & \beta < \gamma, \text{ limit } \beta, f \in \gamma^\beta \end{array}}{\phi_\emptyset \vdash_{\mathbf{x}} \bigvee_{f \in B} \bigwedge_{\beta < \delta_f} \phi_{f|_{\beta+1}}}$$

The exactness property T

Given a locale L, we would like to define points that in addition preserve κ-small intersections. These correspond to κ-complete, completely prime filters of L.

This leads us to a strong distributivity property:

$$\frac{\begin{array}{ll} \phi_f \vdash_{\mathbf{x}} \bigvee_{g \in \gamma^{\beta+1}, g|_\beta = f} \phi_g & \beta < \gamma, f \in \gamma^\beta \\ \phi_f \dashv\vdash_{\mathbf{x}} \bigwedge_{\alpha<\beta} \phi_{f|_\alpha} & \beta < \gamma, \text{ limit } \beta, f \in \gamma^\beta \end{array}}{\phi_\emptyset \vdash_{\mathbf{x}} \bigvee_{f \in B} \bigwedge_{\beta < \delta_f} \phi_{f|_{\beta+1}}}$$

This distributivity property implies κ-distributivity:

$$\bigwedge_{i<\gamma} \bigvee_{j<\gamma} a_{ij} \to \bigvee_{f \in \gamma^\gamma} \bigwedge_{i<\gamma} a_{if(i)}$$

for each $\gamma < \kappa$. But it is strictly stronger than this.

The exactness property T

Given a locale L, we would like to define points that in addition preserve κ-small intersections. These correspond to κ-complete, completely prime filters of L.

This leads us to a strong distributivity property:

$$\frac{\phi_f \vdash_{\mathbf{x}} \bigvee_{g \in \gamma^{\beta+1}, g|_\beta = f} \phi_g \qquad \beta < \gamma, f \in \gamma^\beta \qquad\qquad \phi_f \dashv\vdash_{\mathbf{x}} \bigwedge_{\alpha<\beta} \phi_{f|_\alpha} \qquad \beta < \gamma, \text{ limit } \beta, f \in \gamma^\beta}{\phi_\emptyset \vdash_{\mathbf{x}} \bigvee_{f \in B} \bigwedge_{\beta < \delta_f} \phi_{f|_{\beta+1}}}$$

This distributivity property implies κ-distributivity:

$$\bigwedge_{i<\gamma} \bigvee_{j<\gamma} a_{ij} \to \bigvee_{f \in \gamma^\gamma} \bigwedge_{i<\gamma} a_{if(i)}$$

for each $\gamma < \kappa$. But it is strictly stronger than this. Example: $([0,1], \leq)$

The exactness property T

The exactness property T

The strong distributivity property is the missing rule of inference: although there are geometric theories which are consistent and have no models, the addition of the distributivity property helps to derive $\top \vdash \bot$.

The exactness property T

The strong distributivity property is the missing rule of inference: although there are geometric theories which are consistent and have no models, the addition of the distributivity property helps to derive $\top \vdash \bot$.
Example: given an uncountable set A, the theory of injections $f : A \to \mathbf{N}$ does not have models:

The exactness property T

The strong distributivity property is the missing rule of inference: although there are geometric theories which are consistent and have no models, the addition of the distributivity property helps to derive $\top \vdash \bot$.
Example: given an uncountable set A, the theory of injections $f : A \to \mathbf{N}$ does not have models:

$$P_{a,n} \wedge P_{a,m} \vdash \bot \text{ for all } n, m, a \text{ with } n \neq m$$

$$P_{a,n} \wedge P_{b,n} \vdash \bot \text{ for all } n, a, b \text{ with } a \neq b$$

$$\top \vdash \bigvee_{n \in \mathbf{N}} P_{a,n} \text{ for all } a \in A$$

The exactness property T

The strong distributivity property is the missing rule of inference: although there are geometric theories which are consistent and have no models, the addition of the distributivity property helps to derive $\top \vdash \bot$.

Example: given an uncountable set A, the theory of injections $f : A \to \mathbf{N}$ does not have models:

$$P_{a,n} \wedge P_{a,m} \vdash \bot \text{ for all } n, m, a \text{ with } n \neq m$$

$$P_{a,n} \wedge P_{b,n} \vdash \bot \text{ for all } n, a, b \text{ with } a \neq b$$

$$\top \vdash \bigvee_{n \in \mathbf{N}} P_{a,n} \text{ for all } a \in A$$

As a geometric theory, it is consistent. However, considered as a κ-geometric propositional theory, it is inconsistent.

The exactness property T

The exactness property T

Consider a κ-chain in a category $\mathcal{C}$ with κ-limits, i.e., a diagram $\Gamma : \gamma^{op} \to \mathcal{C}$ specified by morphisms $(h_{\beta,\alpha} : C_\beta \to C_\alpha)_{\alpha \leq \beta < \gamma}$ with the following condition:

The exactness property T

Consider a κ-chain in a category $\mathcal{C}$ with κ-limits, i.e., a diagram $\Gamma : \gamma^{op} \to \mathcal{C}$ specified by morphisms $(h_{\beta,\alpha} : C_\beta \to C_\alpha)_{\alpha \leq \beta < \gamma}$ with the following condition:

- the restriction $\Gamma|_\beta$ is a limit diagram for every limit ordinal β.

The exactness property T

Consider a κ-chain in a category $\mathcal{C}$ with κ-limits, i.e., a diagram $\Gamma : \gamma^{op} \to \mathcal{C}$ specified by morphisms $(h_{\beta,\alpha} : C_\beta \to C_\alpha)_{\alpha \leq \beta < \gamma}$ with the following condition:

- the restriction $\Gamma|_\beta$ is a limit diagram for every limit ordinal β.

We say that the morphisms $h_{\beta,\alpha}$ compose transfinitely, and take the limit projection $f_{\beta,0}$ to be the transfinite composite of $h_{\alpha+1,\alpha}$ for $\alpha < \beta$.

The exactness property T

Consider a κ-chain in a category $\mathcal{C}$ with κ-limits, i.e., a diagram $\Gamma : \gamma^{op} \to \mathcal{C}$ specified by morphisms $(h_{\beta,\alpha} : C_\beta \to C_\alpha)_{\alpha \leq \beta < \gamma}$ with the following condition:

- the restriction $\Gamma|_\beta$ is a limit diagram for every limit ordinal β.

We say that the morphisms $h_{\beta,\alpha}$ compose transfinitely, and take the limit projection $f_{\beta,0}$ to be the transfinite composite of $h_{\alpha+1,\alpha}$ for $\alpha < \beta$. There is an exactness condition on $\mathcal{C}$ that we call *transfinite transitivity*: if we have a κ-tree of morphisms of $\mathcal{C}$ where the immediate successors of every node form a jointly covering family, then the transfinite composites of the morphisms along all possible cofinal branches of the tree forms itself a jointly covering family.

The exactness property T

We have now:

The exactness property T

We have now:

Definition

A κ-geometric category is a category with κ-small limits, complete subobject lattices with stable unions and that satisfies the property T (transfinite transitivity property), i.e., the transfinite composites up to κ of jointly covering families of morphisms is itself jointly covering.

The exactness property T

We have now:

Definition

A κ-geometric category is a category with κ-small limits, complete subobject lattices with stable unions and that satisfies the property T (transfinite transitivity property), i.e., the transfinite composites up to κ of jointly covering families of morphisms is itself jointly covering.

The property T corresponds to the logical rule of inference:

$$\frac{\begin{array}{ll} \phi_f \vdash_{\mathbf{y}_f} \bigvee_{g \in \gamma^{\beta+1}, g|_\beta = f} \exists \mathbf{x}_g \phi_g & \beta < \kappa, f \in \gamma^\beta \\ \phi_f \dashv\vdash_{\mathbf{y}_f} \bigwedge_{\alpha < \beta} \phi_{f|_\alpha} & \beta < \kappa, \text{ limit } \beta, f \in \gamma^\beta \end{array}}{\phi_\emptyset \vdash_{\mathbf{y}_\emptyset} \bigvee_{f \in B} \exists_{\beta < \delta_f} \mathbf{x}_{f|_{\beta+1}} \bigwedge_{\beta < \delta_f} \phi_{f|_{\beta+1}}}$$

κ-geometric categories

κ-geometric categories

Examples of κ-geometric categories:

κ-geometric categories

Examples of κ-geometric categories:

- $\mathcal{S}et$

κ-geometric categories

Examples of κ-geometric categories:

- $\mathcal{S}et$
- Presheaf toposes

κ-geometric categories

Examples of κ-geometric categories:

- $\mathcal{S}et$
- Presheaf toposes
- Sheaves on a site with a κ-Grothendieck topology: site on a category with κ-small limits and where transfinite composites of covering families are covering

κ-geometric categories

Examples of κ-geometric categories:

- $\mathcal{S}et$
- Presheaf toposes
- Sheaves on a site with a κ-Grothendieck topology: site on a category with κ-small limits and where transfinite composites of covering families are covering
- The syntactic category of any κ-geometric theory

κ-geometric categories

Examples of κ-geometric categories:

- $\mathcal{S}et$
- Presheaf toposes
- Sheaves on a site with a κ-Grothendieck topology: site on a category with κ-small limits and where transfinite composites of covering families are covering
- The syntactic category of any κ-geometric theory

There is a way of generating enough examples of κ-separable toposes:

- Start with a category of size at most κ with κ-small limits

κ-geometric categories

Examples of κ-geometric categories:

- $\mathcal{S}et$
- Presheaf toposes
- Sheaves on a site with a κ-Grothendieck topology: site on a category with κ-small limits and where transfinite composites of covering families are covering
- The syntactic category of any κ-geometric theory

There is a way of generating enough examples of κ-separable toposes:

- Start with a category of size at most κ with κ-small limits
- Choose κ many families over each object

κ-geometric categories

Examples of κ-geometric categories:

- $\mathcal{S}et$
- Presheaf toposes
- Sheaves on a site with a κ-Grothendieck topology: site on a category with κ-small limits and where transfinite composites of covering families are covering
- The syntactic category of any κ-geometric theory

There is a way of generating enough examples of κ-separable toposes:

- Start with a category of size at most κ with κ-small limits
- Choose κ many families over each object
- Generate a κ-topology using property T as a closure condition on the chosen families (this takes κ^+ iterations)

κ-geometric categories

Examples of κ-geometric categories:

- $\mathcal{S}et$
- Presheaf toposes
- Sheaves on a site with a κ-Grothendieck topology: site on a category with κ-small limits and where transfinite composites of covering families are covering
- The syntactic category of any κ-geometric theory

There is a way of generating enough examples of κ-separable toposes:

- Start with a category of size at most κ with κ-small limits
- Choose κ many families over each object
- Generate a κ-topology using property T as a closure condition on the chosen families (this takes κ^+ iterations)

κ-geometric categories

Examples of κ-geometric categories:

- $\mathcal{S}et$
- Presheaf toposes
- Sheaves on a site with a κ-Grothendieck topology: site on a category with κ-small limits and where transfinite composites of covering families are covering
- The syntactic category of any κ-geometric theory

There is a way of generating enough examples of κ-separable toposes:

- Start with a category of size at most κ with κ-small limits
- Choose κ many families over each object
- Generate a κ-topology using property T as a closure condition on the chosen families (this takes κ^+ iterations)

Example/exercise: Any Grothendieck topology is an ω-topology (property T is trivially satisfied).

Main results and applications

Main results and applications

- κ-separable toposes have enough κ-points

Main results and applications

- κ-separable toposes have enough κ-points
- Its associated completeness theorem for $\mathcal{L}_{\kappa^+,\kappa}$ is best possible in terms of the cardinalities of the theories involved

Main results and applications

- κ-separable toposes have enough κ-points
- Its associated completeness theorem for $\mathcal{L}_{\kappa^+,\kappa}$ is best possible in terms of the cardinalities of the theories involved
- They coincide with the κ-classifying toposes of κ-geometric theories with at most κ many axioms in canonical form

Main results and applications

- κ-separable toposes have enough κ-points
- Its associated completeness theorem for $\mathcal{L}_{\kappa^+,\kappa}$ is best possible in terms of the cardinalities of the theories involved
- They coincide with the κ-classifying toposes of κ-geometric theories with at most κ many axioms in canonical form
- When κ is weakly compact, this is precisely κ-Deligne's theorem. It is essentially equivalent to Karp's completeness theorem for $\mathcal{L}_{\kappa,\kappa}$

Main results and applications

Main results and applications

- When κ is weakly compact, the completeness of κ-coherent theories allows to generalize to the infinitary case Joyal's completeness theorem for intuitionistic first-order logic

Main results and applications

- When κ is weakly compact, the completeness of κ-coherent theories allows to generalize to the infinitary case Joyal's completeness theorem for intuitionistic first-order logic
- When κ is weakly compact, Stone type dualities for distributive lattices can be generalized to the infinitary case obtaining as a result completeness of infinitary intuitionistic first-order logic over $\mathcal{L}_{\kappa^+,\kappa}$

Main results and applications

- When κ is weakly compact, the completeness of κ-coherent theories allows to generalize to the infinitary case Joyal's completeness theorem for intuitionistic first-order logic
- When κ is weakly compact, Stone type dualities for distributive lattices can be generalized to the infinitary case obtaining as a result completeness of infinitary intuitionistic first-order logic over $\mathcal{L}_{\kappa^+,\kappa}$
- When κ is strongly compact, the completeness theorem provides a generalization to the infinitary case of Makkai's Stone duality for first-order logic, under suitable large cardinal assumptions

Main results and applications

- When κ is weakly compact, the completeness of κ-coherent theories allows to generalize to the infinitary case Joyal's completeness theorem for intuitionistic first-order logic
- When κ is weakly compact, Stone type dualities for distributive lattices can be generalized to the infinitary case obtaining as a result completeness of infinitary intuitionistic first-order logic over $\mathcal{L}_{\kappa^+,\kappa}$
- When κ is strongly compact, the completeness theorem provides a generalization to the infinitary case of Makkai's Stone duality for first-order logic, under suitable large cardinal assumptions
- Using Stone duality for infinitary first-order logic it is possible to prove a descent theorem for κ-pretoposes. This entails both a local and a global definability theorem for infinitary logic

Main results and applications

- When κ is weakly compact, the completeness of κ-coherent theories allows to generalize to the infinitary case Joyal's completeness theorem for intuitionistic first-order logic
- When κ is weakly compact, Stone type dualities for distributive lattices can be generalized to the infinitary case obtaining as a result completeness of infinitary intuitionistic first-order logic over $\mathcal{L}_{\kappa^+,\kappa}$
- When κ is strongly compact, the completeness theorem provides a generalization to the infinitary case of Makkai's Stone duality for first-order logic, under suitable large cardinal assumptions
- Using Stone duality for infinitary first-order logic it is possible to prove a descent theorem for κ-pretoposes. This entails both a local and a global definability theorem for infinitary logic
- The κ-classifying topos allows to generalize to the infinitary case Awodey-Forssell duality for decidable κ-coherent categories

Main results and applications

- When κ is weakly compact, the completeness of κ-coherent theories allows to generalize to the infinitary case Joyal's completeness theorem for intuitionistic first-order logic
- When κ is weakly compact, Stone type dualities for distributive lattices can be generalized to the infinitary case obtaining as a result completeness of infinitary intuitionistic first-order logic over $\mathcal{L}_{\kappa^+,\kappa}$
- When κ is strongly compact, the completeness theorem provides a generalization to the infinitary case of Makkai's Stone duality for first-order logic, under suitable large cardinal assumptions
- Using Stone duality for infinitary first-order logic it is possible to prove a descent theorem for κ-pretoposes. This entails both a local and a global definability theorem for infinitary logic
- The κ-classifying topos allows to generalize to the infinitary case Awodey-Forssell duality for decidable κ-coherent categories

Thank you!