Fibrations of toposes from extensions of theories

*Toposes in Como*

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Idea

Johnstone fibrations in 2-categories

Fibrations of toposes from extension of theories

References
For many special constructions of topological spaces, a structure preserving morphism between the presenting structures gives a map between the corresponding spaces. e.g.: a homomorphism \( f : K \to L \) between two distributive lattices gives a map in the opposite direction between their spectra. The covariance or contravariance of this correspondence is a fundamental property of the construction.
For many special constructions of topological spaces, a structure preserving morphism between the presenting structures gives a map between the corresponding spaces. e.g.: a homomorphism \( f : K \to L \) between two distributive lattices gives a map in the opposite direction between their spectra. The covariance or contravariance of this correspondence is a fundamental property of the construction.

In topos theory we can relativize this process: a presenting structure in an elementary topos \( \mathcal{E} \) will give rise to a bounded geometric morphism \( p : \mathcal{F} \to \mathcal{E} \), where \( \mathcal{F} \) is the topos of sheaves over \( \mathcal{E} \) for the space presented by the structure. Then we commonly find that the covariant or contravariant correspondence mentioned above makes every such \( p \) an opfibration or fibration in the 2-category of toposes and geometric morphisms.

References


Using the classifying toposes of geometric theories, we formalize this idea by the notion of fibration of toposes.
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Of toposes and geometric morphisms.

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For many special constructions of topological spaces, a structure preserving morphism between the presenting structures gives a map between the corresponding spaces. E.g.: a homomorphism $f : K \to L$ between two distributive lattices gives a map in the opposite direction between their spectra. The covariance or contravariance of this correspondence is a fundamental property of the construction.

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Using the classifying toposes of geometric theories, we formalize this idea by the notion of fibration of toposes.
Johnstone fibrations in 2-categories
Comprehension 2-category

Suppose \( \mathbf{K} \) is a 2-category and \( \mathcal{D} \) is a class of bicarrable 1-cells in \( \mathbf{K} \) which we shall call “display 1-cells”. We form a 2-category \( \mathbf{K}_\mathcal{D} \) whose

- 0-cells are of the form

\[
\begin{array}{c}
\bar{x} \\
\downarrow \\
x \\
\downarrow \\
x
\end{array}
\]

where \( x \) is a member of class \( \mathcal{D} \).
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\downarrow x \\
\bar{x}
\end{array}
$$

where $x$ is a member of class $\mathcal{D}$.

- 1-cells from $y$ to $x$ are of the form $f = \langle \bar{f}, f, \bar{f} \rangle$

$$
\begin{array}{c}
\bar{y} \xrightarrow{\bar{f}} \bar{x} \\
\downarrow \downarrow \downarrow \\
y \xrightarrow{f} x
\end{array}
$$

where $f : x \circ \bar{f} \Rightarrow \bar{f} \circ y$ is an iso 2-cell in $\mathbb{K}$. 

Comprehension 2-category
• 2-cells between 1-cells $f$ and $g$ are of the form $\alpha = \langle \bar{\alpha}, \alpha \rangle$ where $\bar{\alpha} : \bar{f} \Rightarrow \bar{g}$ and $\alpha : f \Rightarrow g$ are 2-cells in $\mathcal{K}$

\[
\begin{array}{cccc}
\bar{y} & \bar{g} & \bar{x} \\
\downarrow & \bar{\alpha}/ & \uparrow \\
y & \bar{f} & g & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\bar{y} & f & \bar{x} \\
\end{array}
\]

in such a way that the obvious diagram of 2-cells commutes.

• Composition: by pasting
$K_D$ is a sub 2-category of $K^\bot$ and the following diagram of 2-functors commutes.

$$
\begin{array}{ccc}
K_D & & K^\bot \\
& Base & Cod \\
\downarrow & & \downarrow \\
K & & Cod
\end{array}
$$
**Definition (P. Johnstone, 93)**

Suppose $\mathbb{K}$ is a 2-category. A 1-cell $p: E \to B$ is an (internal) **fibration** in $\mathbb{K}$ if it is bicarrable and for any 2-cell $\alpha: f \Rightarrow g: A \Rightarrow B$ in $\mathbb{K}$, there exists a 1-cell $r(\alpha): g^*E \to f^*E$, a 2-cell $\tilde{\alpha}: p^*f \circ r(\alpha) \Rightarrow p^*g$, and a 2-cell $\tau(\alpha): f^*p \circ r(\alpha) \Rightarrow g^*p$ satisfying **five axioms**.

![Diagram of fibrations in 2-categories](attachment:image.png)
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Suppose $\mathbb{K}$ is a 2-category. A 1-cell $p: E \to B$ is an (internal) **fibration** in $\mathbb{K}$ if it is bicarrable and for any 2-cell $\alpha: f \Rightarrow g: A \Rightarrow B$ in $\mathbb{K}$, there exists a 1-cell $r(\alpha): g^*E \to f^*E$, a 2-cell $\tilde{\alpha}: p^*f \circ r(\alpha) \Rightarrow p^*g$, and a 2-cell $\tau(\alpha): f^*p \circ r(\alpha) \Rightarrow g^*p$ satisfying **five axioms**.
Johnstone’s fibrations in 2-categories

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Peter Johnstone. “Fibrations and partial products in a 2-category”. In: *Applied Categorical Structures* vol.1.2 (June 1993), pp. 141–179. DOI: 10.1007/BF00880041
Remark

- This definition generalizes the definition of Grothendieck fibration of categories.
- The definition above is equivalent to the representable definition of fibration internal to a 2-category.
- Dually, opfibrations are defined by requiring a 1-cell $l(\alpha) : f^* E \to g^* E$ in the opposite direction of $r(\alpha)$.
- Johnstone's definition does not require strictness of the 2-category nor the existence of the structure of strict pullbacks and comma objects. Indeed, this definition is most suitable for weak 2-categories such as 2-category of toposes where we do not expect diagrams of 1-cells to commute strictly. This definition is also very flexible in terms of existence of bipullbacks.
Changing the notation ...

\[ \begin{array}{ccc}
  f^*E & \xrightarrow{p^*g} & E \\
  \downarrow{p^*f} & & \downarrow{p} \\
  A & \xrightarrow{g} & B \\
  \downarrow{\alpha} & & \downarrow{} \\
  A & \xrightarrow{f} & B
\end{array} \]

\[ \begin{array}{ccc}
  \bar{X}g & \xrightarrow{\bar{g}} & \bar{X} \\
  \downarrow{\bar{r}_\alpha} & & \downarrow{} \\
  \bar{X}f & \xrightarrow{\bar{\alpha}} & \bar{X} \\
  \downarrow{} & & \downarrow{} \\
  Y & \xrightarrow{f} & X \\
  \downarrow{} & & \downarrow{} \\
  x & \xrightarrow{} & X
\end{array} \]
Simplifying Johnstone’s definition
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\[
\begin{array}{c}
\text{Base} \\
\downarrow \\
\K \\
\downarrow \\
\K_D
\end{array}
\quad
\begin{array}{c}
x_f \\
\downarrow^f \\
x_g \\
\downarrow^{r_\alpha} \\
x \\
\downarrow^g \\
x
\end{array}
\quad
\begin{array}{c}
y \\
\downarrow^\alpha \\
y \\
\downarrow^f \\
x \\
\downarrow^g \\
x
\end{array}
\]
Simplifying Johnstone’s definition
Axioms of Johnstone fibration

1 $\alpha$ lies over $\alpha$. 

Axioms of Johnstone fibration

2 For any two composable 2-cells $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ in $\mathcal{K}$ where $f, g, h: y \to x$, the lift of their composition is canonically isomorphic to composition of their lifts $\alpha$ and $\beta$ in $\mathcal{K}_D$, that is there exists a vertical iso 2-cell $\tau_{\alpha,\beta} : r_\alpha \circ r_\beta \Rightarrow r_{\beta\alpha}$ in $\mathcal{K}_D$ such that $\beta \circ (\alpha \cdot r_\beta) \circ (f \cdot \tau_{\alpha,\beta}^{-1})$ is the lift of $\beta \circ \alpha$. 

\[
\begin{array}{ccc}
x_h & \xrightarrow{r_\beta} & x_g \\
\downarrow & & \downarrow \\
r_{\beta\alpha} & \xleftarrow{\tau_{\alpha,\beta}} & x_f \\
\downarrow & & \downarrow \\
x & \xrightarrow{f} & x
\end{array}
\]
Axioms of Johnstone fibration

3. For any 1-cell $f : y \to x$ the lift of identity 2-cell on $f$ is canonically isomorphic to the identity 2-cell on the lift $f$, that is there exists a vertical iso 2-cell $\tau_f : 1_f \Rightarrow r_{id_f}$ in $\mathcal{K}_D$ such that $f \cdot \tau_f^{-1}$ is the lift of identity 2-cell $id_f$. 

\[
\begin{array}{ccc}
X_f & \xrightarrow{\tau_f} & 1_f \\
\downarrow & & \downarrow \\
X_f & \xleftarrow{r_{id_f}} & X_f
\end{array}
\quad
f
\quad
X_f \xrightarrow{f} X
\]
Axioms of Johnstone fibration

4 lift of whiskering of any 2-cell \( \alpha: f \to g: y \to x \) with any 1-cell \( k: z \to y \) is isomorphic, via vertical iso 2-cells \( \gamma \) and \( \gamma' \), to whiskering of the lifts.
\textbf{Axioms of Johnstone fibration}

5 Given any pair of vertical 1-cells $v_0 : y \to x_f$ and $v_1 : y \to x_g$, any 2-cell $\gamma : f \circ v_0 \Rightarrow g \circ v_1$ factors through $\alpha$ uniquely, that is there exists a unique 2-cell $\mu : v_0 \Rightarrow r_\alpha v_1$ such that the following pasting diagrams are equal.

\[ 
\begin{array}{ccc}
  y & \xrightarrow{v_0} & x_f \\
  \downarrow v_1 & \Downarrow \gamma & \downarrow f \\
  x_g & \xrightarrow{g} & x
\end{array} 
= 
\begin{array}{ccc}
  y & \xrightarrow{v_0} & x_f \\
  \downarrow v_1 & \Downarrow \mu \downarrow & \downarrow f \\
  x_g & \xrightarrow{r_\alpha} & x
\end{array} 
\]
(Weak) cartesian 1-cells

**Definition**

Suppose $P : X \to C$ is a 2-functor. A 1-cell $f : X \to Y$ in $X$ is **cartesian** with respect to $P$ whenever for each 0-cell $W$ in $X$ the following commuting square is a *bipullback* diagram in 2-category $\text{Cat}$ of categories.

$$
\begin{array}{ccc}
X(W, X) & \xrightarrow{f_*} & X(W, Y) \\
\downarrow_{P_{W,X}} & & \downarrow_{P_{W,Y}} \\
C(PW, PX) & \xrightarrow{P(f)_*} & C(PW, PY)
\end{array}
$$
Cartesian 1-cells in elementary terms

**Input data:**
1. \( g : W \to Y \)
2. \( h : PW \to PX \)
3. iso 2-cell \( \alpha : P(f) \circ h \Rightarrow P(g) \)

**Output data:**
(*not necc. unique*)
1. \( \hat{h} : W \to X \)
2. iso 2-cell \( \hat{\alpha} : f \hat{h} \Rightarrow g \)
3. iso 2-cell \( \hat{\beta} : P(\hat{h}) \Rightarrow h \)
4. an equality of 2-cells
   \[ \alpha \circ (P(f) \cdot \hat{\beta}) = P(\hat{\alpha}) \]
Idea

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References

\[
\begin{align*}
\text{Input data:} & \\
1 & \sigma : g \Rightarrow g' : W \Rightarrow Y \\
2 & \delta : h \Rightarrow h' : PW \Rightarrow PX \\
3 & \text{iso 2-cells} \\
& \alpha : P(f) \circ h \Rightarrow P(g) \\
& \alpha' : P(f) \circ h' \Rightarrow P(g) \\
4 & \text{an equality of 2-cells} \\
& \alpha' \circ (Pf \cdot \delta) = P(\sigma) \circ \alpha
\end{align*}
\]

\[
\begin{align*}
\text{Output data:} & \\
1 & \text{unique } \hat{\delta} : \hat{h} \Rightarrow \hat{h}' \\
2 & \text{an equality } \hat{\alpha}' \circ (f \cdot \hat{\delta}) = \sigma \circ \hat{\alpha} \\
3 & \text{an equality } \delta \cdot (\hat{\beta}) = \hat{\beta}' \circ P\delta
\end{align*}
\]
**Definition**

A 2-cell $\alpha : f \Rightarrow g : x \to y$ in $\mathbb{X}$ is **cartesian** if it is cartesian as a 1-cell for the functor $P_{xy} : \mathbb{X}(x, y) \to \mathbb{C}(P_x, P_y)$. 
**Definition**

A 2-cell $\alpha : f \Rightarrow g : x \to y$ in $\mathbf{X}$ is **cartesian** if it is cartesian as a 1-cell for the functor $\mathcal{P}_{xy} : \mathbf{X}(x, y) \to \mathbf{C}(\mathcal{P}x, \mathcal{P}y)$.

In elementary terms it means a 2-cell $\alpha : f \Rightarrow g : X \Rightarrow Y$ is cartesian if for any given 1-cell $e : X \to Y$ and 2-cell $\beta : e \Rightarrow g$ with $\mathcal{P}\alpha = \mathcal{P}\beta \circ \gamma$ for some 2-cell $\gamma$, then there is a unique 2-cell $\tilde{\gamma}$ over $\gamma$ such that $\alpha = \beta \circ \tilde{\gamma}$.
**Proposition**

A 1-cell $x : \overline{x} \to \overline{x}$ in $K$ is a Johnstone fibration iff

1. every $f : y \to x = \text{Cod}(x)$ has a cartesian lift,
2. for every 0-cell $y$ in $K_D$, the functor
   $$\text{Cod}_{y,x} : K_D(y, x) \to K(\text{Cod}(y), \text{Cod}(x))$$
   is a Grothendieck fibration of categories, and
3. whiskering on the left preserves cartesian 2-cells in $K_D$ between 1-cells with codomain $x$. 
Relating internal fibrations in 2-categories to fibration of bicategories

**Definition**

Let $P : X \to C$ be a 2-functor. $X$ is **fibred over** $C$ whenever

1. for any $X \in X$ and $f : B \to PX$ in $C$, there is a weakly cartesian 1-cell $\tilde{f} : \tilde{B} \to X$ with $P\tilde{f} = f$;
2. $P$ is locally fibred, i.e. $P_{XY} : X(X, Y) \to C(PX, PY)$ is a Grothendieck fibration of categories for all $X, Y$ in $X$
3. The horizontal composite of any two cartesian 2-cells is again cartesian.
Relating internal fibrations in 2-categories to fibration of bicategories

**Definition**

Let $\mathcal{P}: \mathcal{X} \to \mathcal{C}$ be a 2-functor. $\mathcal{X}$ is **fibred over** $\mathcal{C}$ whenever

1. for any $X \in \mathcal{X}$ and $f: B \to \mathcal{P}X$ in $\mathcal{C}$, there is a weakly cartesian 1-cell $\tilde{f}: \tilde{B} \to X$ with $\mathcal{P}\tilde{f} = f$;
2. $\mathcal{P}$ is locally fibred, i.e. $P_{XY}: \mathcal{X}(X, Y) \to \mathcal{C}(\mathcal{P}X, \mathcal{P}Y)$ is a Grothendieck fibration of categories for all $X, Y$ in $\mathcal{X}$
3. The horizontal composite of any two cartesian 2-cells is again cartesian.

This definition is due to Buckley 2014 and he develops a theory of fibred bicategories in Mitchell Buckley. “Fibred 2-categories and bicategories”. In: vol. 218 (2014), pp. 1034–1074

The theory of fibred bicategories was also independently developed by Bakovic 2012 intrinsically to tricategories in Igor Bakovic. “Fibrations in tricategories”. In: 93rd Peripatetic Seminar on Sheaves and Logic, University of Cambridge (2012)
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**Remark**

$K_D$ is fibred over $K$ if every 1-cell in $K_D$ is a fibration in the sense of Johnstone.
2-categories (really bicategories) of toposes

- The 2-category $\mathcal{E}\mathsf{Top}$ is the 2-category of elementary toposes, geometric morphisms, and natural transformations.
- The 2-category $\mathcal{G}\mathsf{Top}$ is constructed from 2-category $\mathcal{E}\mathsf{Top}$ by choosing the class of display morphisms to be bounded geometric morphisms of elementary toposes. So, $\mathcal{G}\mathsf{Top} = \mathcal{E}\mathsf{Top}_D$ where $D$ is the class of bounded geometric morphisms of elementary toposes.

\[
\begin{array}{ccc}
\mathcal{G}\mathsf{Top} & \xrightarrow{} & \mathcal{E}\mathsf{Top} \\
\downarrow & & \downarrow \\
\mathcal{E}\mathsf{Top} & \xleftarrow{} & \mathcal{G}\mathsf{Top}
\end{array}
\]

- A bounded geometric morphism $p: \mathcal{E} \to \mathcal{I}$ is a fibration of toposes if it is a fibration 0-cell in $\mathcal{G}\mathsf{Top}$. 
Classifying toposes as representing objects

- Consider the pseudofunctor

\[
\mathcal{T}\text{-Mod} : (\mathcal{B}\text{Top}/S)^{\text{op}} \to \mathcal{C}at
\]
Classifying toposes as representing objects

- Consider the pseudofunctor

\[ \mathbb{T}\text{-Mod} : (\mathcal{B}\text{Top}/S)^{op} \to \text{Cat} \]

- To an \( \mathcal{S} \)-topos \( \mathcal{E} \) it assigns the category \( \mathbb{T}\text{-Mod}\mathcal{E} \) of models \( \mathbb{T} \) in \( \mathcal{E} \).
Classifying toposes as representing objects

- Consider the pseudofunctor

\[ \mathbb{T}\text{-Mod-}: (B\mathcal{Top}/S)^{op} \to \mathcal{Cat} \]

- To an \( \mathcal{J} \)-topos \( \mathcal{E} \) it assigns the category \( \mathbb{T}\text{-Mod-} \mathcal{E} \) of models \( \mathbb{T} \) in \( \mathcal{E} \).

- To a geometric morphism \( \langle f^*, f_* \rangle: \mathcal{F} \to \mathcal{E} \) of \( \mathcal{J} \)-toposes it assigns the functor \( f^*: \mathbb{T}\text{-Mod-} \mathcal{E} \to \mathbb{T}\text{-Mod-} \mathcal{F} \).
Classifying toposes as representing objects

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• To a geometric morphism \( \langle f^*, f_* \rangle : \mathcal{F} \to \mathcal{E} \) of \( \mathcal{S}\)-toposes it assigns the functor \( f^* : \mathbb{T}\text{-Mod}\mathcal{E} \to \mathbb{T}\text{-Mod}\mathcal{F} \).

• Note that \( \mathbb{T}\text{-Mod} (f \circ g) \cong (\mathbb{T}\text{-Mod} f) \circ (\mathbb{T}\text{-Mod} g) \)
Classifying toposes as representing objects

• Consider the pseudofunctor

\[ \mathbb{T}\text{-Mod-} : (\mathcal{B}\mathcal{T}\text{op}/S)^{\text{op}} \to \text{Cat} \]

• To an \(\mathcal{I}\)-topos \(\mathcal{E}\) it assigns the category \(\mathbb{T}\text{-Mod-} \mathcal{E}\) of models \(\mathbb{T}\) in \(\mathcal{E}\).

• To a geometric morphism \(\langle f^*, f_* \rangle : \mathcal{F} \to \mathcal{E}\) of \(\mathcal{I}\)-toposes it assigns the functor \(f^* : \mathbb{T}\text{-Mod-} \mathcal{E} \to \mathbb{T}\text{-Mod-} \mathcal{F}\).

• Note that \(\mathbb{T}\text{-Mod-} (f \circ g) \cong (\mathbb{T}\text{-Mod-} f) \circ (\mathbb{T}\text{-Mod-} g)\)

• The classifying topos \(\mathcal{I}[\mathbb{T}]\) of a geometric theory/context \(\mathbb{T}\) can be seen as a representing object for this pseudofunctor, i.e.

\[ \mathcal{B}\mathcal{T}\text{op}/\mathcal{I}(\mathcal{E}, \mathcal{I}[\mathbb{T}]) \cong \mathbb{T}\text{-Mod-} \mathcal{E} \]

naturally in \(\mathcal{E}\).
Fibrations of toposes from extension of theories
• Fix an elementary topos $S$. Every geometric theory/context $T$ gives rise to an indexed category over $\mathbf{T} : \mathcal{B} \text{Top}/S$, where

$$\mathcal{T}(\mathcal{E}): = \mathbf{T-Mod}(\mathcal{E}) = \text{category of models of } T \text{ in } \mathcal{E}$$
• Fix an elementary topos $S$. Every geometric theory/context $T$ gives rise to an indexed category over $T : \mathcal{B} \mathcal{T} \mathcal{op}/S$, where

$$\mathcal{T}(\mathcal{E}) : = T - \text{Mod}-(\mathcal{E}) = \text{category of models of } T \text{ in } \mathcal{E}$$

• Note that $\mathcal{T}$ encapsulates data of all the models in all Grothendieck toposes (with base $S$). Vickers calls them ”elephant theories” after Johnstone, the sheer size of data encoded by them.


• Fix an elementary topos $S$. Every geometric theory/context $T$ gives rise to an indexed category over $T : B\mathbf{Top}/S$, where

$$\mathbb{T}(E) : = T\text{-Mod}(E) = \text{category of models of } T \text{ in } E$$

• Note that $T$ encapsulates data of all the models in all Grothendieck toposes (with base $S$). Vickers calls them ”elephant theories” after Johnstone, the sheer size of data encoded by them.

• Of course not all elephant theories arise from contexts. For instance, given a bounded geometric morphism $p : E \to \mathcal{S}$ and a context extension $U : T_1 \to T_0$ is a context extension and $M$ is a strict model of context $T$ in base topos $S$, then $T_1/M$ is an elephant theory but not a context, where

$$T_1/M(E) : = \text{strict models of } T_1 \text{ in } E \text{ which reduce to } p^*M \text{ via } U$$
• Fix an elementary topos $S$. Every geometric theory/context $\mathbb{T}$ gives rise to an indexed category over $\mathbb{T} : \mathcal{B} \mathcal{T} \mathcal{op} / S$, where

$$\mathbb{T}(\mathcal{E}) : = \mathbb{T} - \text{Mod} - (\mathcal{E}) = \text{category of models of } \mathbb{T} \text{ in } \mathcal{E}$$

• Note that $\mathbb{T}$ encapsulates data of all the models in all Grothendieck toposes (with base $S$). Vickers calls them ”elephant theories” after Johnstone, the sheer size of data encoded by them.

• Of course not all elephant theories arise from contexts. For instance, given a bounded geometric morphism $p : \mathcal{E} \to \mathcal{S}$ and a context extension $U : \mathbb{T}_1 \to \mathbb{T}_0$ is a context extension and $M$ is a strict model of context $\mathbb{T}$ in base topos $S$, then $\mathbb{T}_1 / M$ is an elephant theory but not a context, where

$$\mathbb{T}_1 / M(\mathcal{E}) : = \text{strict models of } \mathbb{T}_1 \text{ in } \mathcal{E} \text{ which reduce to } p^* M \text{ via } U$$

• Certain elephant theories are geometric and have classifying toposes. $\mathbb{T}$ and $\mathbb{T}_1 / M$ are such examples.
**Theorem (Vickers, 2017)**

Suppose $U : \mathbb{T}_1 \to \mathbb{T}_0$ is a context extension. For any model $M$ of $\mathbb{T}_0$ in a (base) topos $S$, $S[\mathbb{T}_1/M]$ is an $S$-topos, and moreover, for any geometric (not necessarily bounded) morphism $f : A \to S$, the classifying topos $A[\mathbb{T}_1/f^*M]$ is got by bi-pullback of $S[\mathbb{T}_1/M]$ along $f$:

$$
\begin{array}{ccc}
A[\mathbb{T}_1/f^*M] & \xrightarrow{\bar{f}} & S[\mathbb{T}_1/M] \\
\downarrow^{p_f} & & \downarrow^p \\
A & \xrightarrow{f} & S
\end{array}
$$

**Theorem (Vickers, 2017)**

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$$
\begin{align*}
\mathcal{A}[\mathbb{T}_1/f^*M] & \xrightarrow{\overline{f}} S[\mathbb{T}_1/M] \\
p_f & \downarrow \\
\mathcal{A} & \xrightarrow{\overline{f}} S
\end{align*}
$$

Chevalley fibrations

- Suppose $\mathcal{K}$ is a 2-category with finite (strict) PIE-limits, in other words those reducible to Products, Inserters and Equifiers.
- This is enough to guarantee existence of all strict comma objects.
Chevalley fibrations

- Suppose $\mathcal{K}$ is a 2-category with finite (strict) PIE-limits, in other words those reducible to Products, Inserters and Equifiers.
- Suppose $B$ is an object of $\mathcal{K}$, and $p$ is a 0-cell in the strict slice 2-category $\mathcal{K}/B$. $p$ is a **Chevalley fibration** if the 1-cell $\Gamma_1$ has a right adjoint $\Lambda_1$ with counit an identity in the 2-category $\mathcal{K}/B$. 

![Diagram](image-url)
Chevalley fibrations

- Dually one defines Chevalley **opfibrations** as 1-cells $p : E \to B$ for which the morphism $\Gamma_0 : E^\perp \to p/B$ has a left adjoint $\Lambda_0$ with identity unit.
Chevalley fibrations

- Dually one defines Chevalley opfibrations as 1-cells $p: E \to B$ for which the morphism $\Gamma_0: E^\downarrow \to p/B$ has a left adjoint $\Lambda_0$ with identity unit.
- A bifibration is equipped with the structures of both a fibration and an opfibration.
Chevalley fibrations

- Dually one defines Chevalley **opfibrations** as 1-cells \( p : E \to B \) for which the morphism \( \Gamma_0 : E^\bot \to p/B \) has a left adjoint \( \Lambda_0 \) with identity unit.
- A bifibration is equipped with the structures of both a fibration and an opfibration.
- Gray 1966 showed that Chevalley fibrations in the 2-category \( \mathcal{C}at \) of (small) categories correspond to well-known (cloven) Grothendieck fibrations.
Chevalley fibrations

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In the case where $p$ is carrable, the comma objects $p/B$ and $B/p$ can be expressed as pullbacks along the two projections from $B \downarrow \downarrow = B/B$ to $B$. 
Fibrational extensions of contexts

- In the case where \( p \) is carrable, the comma objects \( p/B \) and \( B/p \) can be expressed as pullbacks along the two projections from \( B\downarrow = B/B \) to \( B \).
- Any extension map of contexts \( U: \mathbb{T}_1 \to \mathbb{T}_0 \) in the 2-category \( \text{Con} \) is (strictly) carrable.
Fibrational extensions of contexts

• In the case where \( p \) is carrable, the comma objects \( p/B \) and \( B/p \) can be expressed as pullbacks along the two projections from \( B^\downarrow = B/B \) to \( B \).

• Any extension map of contexts \( U: \mathbb{T}_1 \to \mathbb{T}_0 \) in the 2-category \( \mathcal{C}on \) is (strictly) carrable.

• Using this fact, and since comma objects exists in \( \mathcal{C}on \), we reformulate the notion of Chevalley fibration in \( \mathcal{C}on \).
Fibrational extensions of contexts

- An extension map is called **fibrational** if $\Gamma_1$ has a right adjoint with identity counit.
Main theorem

**Theorem**

If $U : \mathbb{T}_1 \to \mathbb{T}_0$ is a (op)fibrational extension of contexts, and $M$ is any model of $\mathbb{T}_0$ in an elementary topos $S$, then $p : S[\mathbb{T}_1/M] \to S$ is an (op)fibration of toposes.
Main theorem

**Theorem**

If $U : T_1 \to T_0$ is a (op)fibrational extension of contexts, and $M$ is any model of $T_0$ in an elementary topos $S$, then $p : S[T_1/M] \to S$ is an (op)fibration of toposes.

Local homeomorphism of toposes as opfibration

• For $S$ a bounded $S_0$ topos, and $T_0 = \emptyset$ and $T_1$ the extended context of $T_0$ with a fresh edge from terminal to the unique node of $T_0$. 
Local homeomorphism of toposes as opfibration

- For $S$ a bounded $S_0$ topos, and $T_0 = \emptyset$ and $T_1$ the extended context of $T_0$ with a fresh edge from terminal to the unique node of $T_0$.
- We get a context extension map $T_1 \to T_0$, which is an opfibration.
Local homeomorphism of toposes as opfibration

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- We get a context extension map $T_1 \to T_0$, which is an opfibration.
- And a bipullback of toposes

$$
\begin{array}{c}
S/M \simeq S[T_1/M] \\
\downarrow \quad M^*p \\
S \\
\downarrow \\
S_0[\mathcal{X}] \\
\end{array}
\xrightarrow{\quad p \quad} 
\begin{array}{c}
S_0[\mathcal{X}, x] = S_0[\mathcal{X}[T_1/\mathcal{X}]] \\
\downarrow \\
S_0[\mathcal{X}] \\
\end{array}
$$

$M^*p$ is a fibration of toposes.
Local homeomorphism of toposes as opfibration

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- We get a context extension map $T_1 \to T_0$, which is an opfibration.
- And a bipullback of toposes

\[
\begin{array}{ccc}
S/M & \xrightarrow{\cong} & S[T_1/M] \\
\downarrow^{M^*p} & & \downarrow^p \\
S & \xrightarrow{M} & S_0[X]
\end{array}
\]

- $M^*p$ is a fibration of toposes.
Spectrum of Boolean algebras

- For $\mathcal{S}$ a bounded $\mathcal{S}_0$ topos, and $\mathbb{T}_0 =$ context of Boolean algebras and $\mathbb{T}_1$ the extended context of Boolean algebra with a prime filter
- We get a context extension map $\mathbb{T}_1 \rightarrow \mathbb{T}_0$ which is a fibration.
- And a bipullback of toposes

$$
\begin{array}{ccc}
Spec(B) & \longrightarrow & S[\mathbb{T}_1/B] \\
\downarrow M^*p & & \downarrow p \\
1 & \longrightarrow & S
\end{array}
$$

- The points of $S[\mathbb{T}_1/B]$ are pairs $(B, F)$ where $F$ is an internal prime filter of $B$ in topos $S$. “every fibrewise Stone bundle is a fibration.”
Other examples

- Internal Algebraic dcpo as opfibrations
- Spectral spaces as fibrations
- SFP domains as bifibrations
- Internal groups equipped with an action as fibrations
- Internal categories equipped with a torsor as opfibrations
- Internal modules as bifibrations
- Bag domains as opfibrations
- ...
Igor Bakovic. “Fibrations in tricategories”. In: 93rd Peripatetic Seminar on Sheaves and Logic, University of Cambridge (2012).


References II


Thank you for your attention!