Elementary toposes

from a predicative point of view

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our motivation

to extend topos theory

to predicative mathematics

our aim

to generalize the axiomatization

of an elementary topos

to include examples of

categories of sets/classes

formalized in predicative foundations

Inspired by the work in this line

started by *I.Moerdijk* and *E.Palmgren*

in their paper "*Wellfounded trees in categories*", Apal 1999,

and later developed by *I.Moerdijk-B. van den Berg*

and also by

"Algebraic Set Theory", *A. Joyal* and *I. Moerdijk*, CUP, 1995

Abstract of our talk

- Motivating background on predicative constructive foundations

- a predicative generalization of the notion of elementary topos

- examples formalizable in predicative foundations of mathematics

Our specific aim

to introduce

a *predicative generalization* of elementary toposes

such that

its generic internal language L_{ptop} is predicative
its generic internal language L_{ptop} is equivalent to that of elementary toposes when a resizing rule/reducibility axiom is added to it
it includes categorical structures of sets/classes of predicative foundations of mathematics as examples

Starting point

The **generic internal language** of elementary toposes

is impredicative in the sense of Russell-Poincarè

what we mean by internal language of elementary toposes

Let ElTop be the category of elementary toposes and logical morphisms.

The internal language of elementary toposes

is provided by a calculus TT_{topoi}

called "**generic internal language**" with the associated category

$\mathrm{Th}(\mathsf{TT}_{\mathbf{topoi}})$

objects	theories of $\mathsf{TT}_{\mathbf{topoi}}$
	(= TT_{topoi} + axioms)
morphisms	translations

together with

---

a functor extracting the **internal theory** out of an elementary topos

$$\begin{array}{rccc} Int: & \mathrm{ElTop} & \rightarrow & \mathsf{Th}(\mathsf{TT}_{topoi}) \\ & \mathcal{E} & \mapsto & Int(\mathcal{E}) \end{array}$$

internal theory of $\mathcal{E}$

---

a functor associating a syntactic elementary topos to a theory $\mathcal{T}$

$$\begin{array}{rccc} Syn: & \mathrm{Th}(\mathsf{TT}_{topoi}) & \rightarrow & \mathrm{ElTop} \\ & T & \mapsto & Syn(T) \end{array}$$

syntactic model of T

---

such that

for any elementary topos E	for any theory T
$E \simeq Syn(Int(E))$	$T \simeq Int(Syn(T))$
equivalent categories	equivalent theories

Internal language *for various other categorical structures*

adapting the previous definition to other categories of categorical structures

internal language as a dependent type theory à la Martin-Löf

is given for the following categorical structures:

lex categories

regular categories

locally cartesian closed categories

pretopoi

arithmetic universes

elementary topoi...

in M.E. M. "Modular correspondence between dependent type theories and categories, MSCS 2005"

Two internal languages of an elementary topos

- **Benabou-Mitchell language**

  formulated as a many-sorted INTUITIONISTIC logic

- **Dependent type theory** à la Martin-Löf

  in

  [M.E.M. PhD thesis'98]

  [M.E.M.'05] *"Modular correspondence between dependent type theories and categorical universes including pretopoi and topoi."* MSCS, 2005

Comparing **Internal languages** *of topoi*

Benabou/Mitchell language	Internal language in [M'05]
many sorted logic with simple types	Dependent type theory with dependent types
sets (=types) $\neq$ propositions	propositions as mono types
propositions= terms of type $\mathcal{P}(1)$	the type $\mathcal{P}(1)$ classifies mono types up to equiprovability

Syntactic topos:

	Benabou/Mitchell language	Internal language in [M'05]
objects	subsets	closed types
morphisms	functional relations	typed terms

Notion of proposition in a topos in [M'05]

In the **generic internal dependent type theory** TT_{topoi} of topoi

a *proposition* P is a *mono type*: if we derive $P\ type$ and a proof p $p \in Eq(P, w, z)\ [w \in P, z \in P]$
a *predicate* $P(x)\ [x \in A]$ is a *mono dependent type*: if we derive $P(x)\ type\ [x \in A]$ and a proof p $p \in Eq(P, w, z)\ [x \in A, w \in P(x), z \in P(x)]$

similar to Homotopy Type Theory

but in an extensional type theory à la Martin-Löf

Impredicativity of the internal language of an elementary topos

let $\mathcal{P}(A)$ be the type of the powerobject of a type A

for any propositional function

$$\phi(x, y)\ [x \in \mathcal{P}(A)\ ,\ y \in A]$$

we can for

$$\{\, y \in A \ \mid\ \forall x \in \mathcal{P}(A)\ \ \phi(x, y)\ \} \varepsilon \mathcal{P}(A)$$

i.e. the power-object is closed under a subset defined by a quantification over all subsets including itself....

Characteristics of predicative definitions

in the sense of *Russell-Poincarè*

Whatever involves an apparent variable must *not be among the possible values* of that variable.

Why being predicative?

for a finer analysis of mathematical concepts and proofs

cfr. *H. Friedman*'s "Reverse mathematics"

Why being predicative and constructive?

for a finer analysis of mathematical concepts and proofs

+

"extractions of programs from proofs"

to perform "constructive reverse mathematics"

what is CONSTRUCTIVE mathematics?

common intended meaning:

CONSTRUCTIVE mathematics
=
mathematics within INTUITIONISTIC LOGIC

what is CONSTRUCTIVE mathematics?

more specific meaning:

constructive mathematics

=

implicit computational mathematics

=

mathematics formalizable in a foundation

which admits a computational interpretation

to view explicitly

constructive proofs as programs	functions between natural numbers as computable

About the plurality of foundations of mathematics

classical mathematics	constructive mathematics
one *standard* impredicative foundation	NO standard foundation
ZFC axiomatic set theory	but different incomparable foundations

Plurality of foundations for matheamtics

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory HoTT and Voevodsky's Univalent Foundations Feferman's constructive expl. maths

Plurality of foundations ⇒ need of a minimalist foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory HoTT and Voevodsky's Univalent Foundations Feferman's constructive expl. maths

what common core ??

Need of a MINIMALIST FOUNDATION

in

[M.E. Maietti & G. Sambin'05] "Toward a minimalist foundation for constructive mathematics"

In L. Crosilla and P. Schuster, eds., From Sets and Types to Topology and Analysis:

Practicable Foundations for Constructive Mathematics, OUP, 2005

we argued:

Plurality of constructive foundations (often mutual incompatible)

$\Downarrow$

need of a core foundation where to find common proofs

to reduce assumptions to a minimum

for ex. to extend reverse mathematics to constructive mathematics

(as advocated by H. Ishihara)

our current notion of constructive foundation for mathematics

j.w.w. G. Sambin

two-level theory	extensional level (used by mathematicians to developed their proofs) ⇓ interpreted via a **QUOTIENT model** intensional level (language suitable for computer-aided formalization of proofs)
extra level	⇓ realizability level (used by computer scientists to extract programs)

Our TWO-LEVEL Minimalist Foundation

from

[M.E.M.'09] "A minimalist two-level foundation for constructive mathematics", Apal, 2009

following

[M.E.M.-G.Sambin'05]

its INTENSIONAL level, called **mtt**

= a PREDICATIVE VERSION of Coquand's Calculus of Constructions (Coq).

= a FRAGMENT of Martin-Löf's intensional type theory + one UNIVERSE

---

its EXTENSIONAL level, called **emtt**

has a PREDICATIVE LOCAL set theory

extension of extensional type theory à la Martin-Löf

+ *effective quotients*

+ *power-collections*

+ *proof-irrelevant propositions*

Plurality of foundations has a Minimalist Foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory HoTT and Voevodsky's Univalent Foundations Feferman's constructive expl. maths

the MINIMALIST FOUNDATION (**MF**) is a common core

We describe some main features of predicative foundations

relevant to our aim

Main features of Aczel's CZF

example of a constructive axiomatic set theory where

the powerclass $\mathcal{P}(A)$ of a set A is NOT a set but a class
NO full separation but limited separation: $\{ x \in A \mid \phi(x) \}$ subset of a set A only when ϕ has bounded quantifiers (i.e. has quantifiers restricted to sets)

ENTITIES in CZF

small propositions $\hookrightarrow$ propositions

sets $\hookrightarrow$ classes

where

small propositions = propositions with restricted quantifiers

as in

"*Algebraic Set Theory*" *A. Joyal - I. Moerdijk*, OUP, 1995

Main features of Martin-Löf's intensional type

characteristics of Martin-Löf's intensional type MTT:

example of a dependent type theory with

NO powerset of a set but only a cumulative hierarchy of universes of sets
all sets can be inductively generated: they can come equipped with an induction principle to define properties on them and a recursion operator to define functions on them

Main features of Homotopy type theory/Voevodsky's Univalent Foundations

characteristics of Homotopy Type Theory (HoTT):

extension of MLTT with

Voevodsky 's Univalence Axiom

higher inductive quotients of sets

Peculiarity of the **Minimalist foundation**

its extension with the principle of excluded middle remains predicative *contrary to* the other predicative foundations Aczel's CZF Martin-Löf's type theory HoTT
functional relations between sets do NOT form a set $\Downarrow$ Dedekind real numbers and Cauchy real numbers do NOT form a set

two notions of function in **MF**

a *primitive notion* of type-theoretic function

$$f(x) \in B \; [x \in A]$$

$\neq$ (syntactically)

notion of functional relation

$$\forall x \in A \; \exists! y \in B \; R(x, y)$$

$\Rightarrow$ NO axiom of unique choice in **MF**

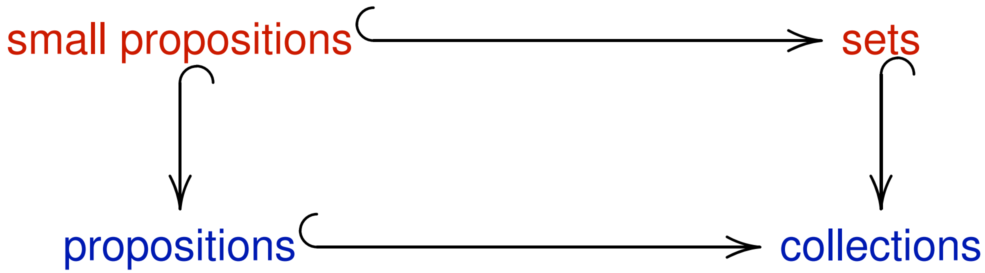
Axiom of unique choice

$$\forall x \in A\, \exists! y \in B\, R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\, \forall x \in A\, R(x, f(x))$$

turns a functional relation into a type-theoretic function.

$\Rightarrow$ identifies the two distinct notions...

ENTITIES in the Minimalist Foundation



Why we need to have both classes/collections and sets

in **MF** and in *Aczel*'s **CZF**

Constructive predicative notion of Locale

=

Formal Topology by *P. Martin-Löf* and *G. Sambin*

represented by the fixpoints of a closure operator

on a base of opens B assumed to be a preorder set:

$$\begin{array}{llll} \mathcal{A}_{\lhd}: & \mathcal{P}(B) & \longrightarrow & \mathcal{P}(B) \\ & U & \mapsto & \{x \in B \mid x \lhd U\} \end{array}$$

satisfying a convergence property:

$$\mathcal{A}_{\lhd}(U \downarrow V) = \mathcal{A}_{\lhd}(U) \cap \mathcal{A}_{\lhd}(V)$$

$$U \downarrow V \equiv \{a \in B \mid \exists u \varepsilon U \, a \leq u \,\&\, \exists v \varepsilon V \, a \leq v\}$$

NO restriction to inductively generated formal topologies

Our specific aim

to introduce

a *predicative generalization* of elementary toposes

such that

---

its generic internal language L_{ptop} is predicative

---

its generic internal language L_{ptop} is equivalent to that of elementary toposes

when a resizing rule/reducibility axiom is added to it

---

it includes categorical structures of sets/classes

of predicative foundations of mathematics

as examples

A further generalization...

we want to predicatively generalize

the notion of arithmetic quasi toposes

i.e. of quasi-toposes with a N.n.o and stable effective quotients of strong eq.rel

its generic internal language L_{p-qtop} is predicative

its generic internal language L_{p-qtop} is equivalent to that of arithmetic quasi-toposes

when a resizing rule/reducibility axiom is added to it

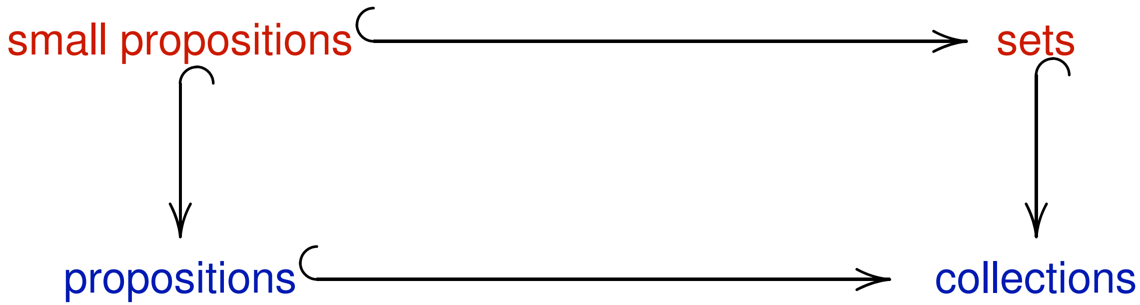
it includes categorical structures of sets/collections

formalized in predicative foundations

including the extensional level of the Minimalist Foundation

in our generalizations

we keep these distinctions



Some notations on fibered categories

By the word fibration we mean a fibred category

$$\mathcal{S} \xrightarrow[p]{} \mathcal{C}$$

such that for any object $\mathcal{S}$-object B and any $\mathcal{C}$-morphism

$$f\colon Y \to p(B)$$

there exists a cartesian arrow u: $A \to B$ over f.

We use the notation

$$cod\colon \mathcal{C}^{\to} \to \mathcal{C}$$

to denote the codomain fibration of a finite limits category $\mathcal{C}$.

Predicative Generalization of Elementary topos

A **predicatively generalized** elementary topos is given by a fibration

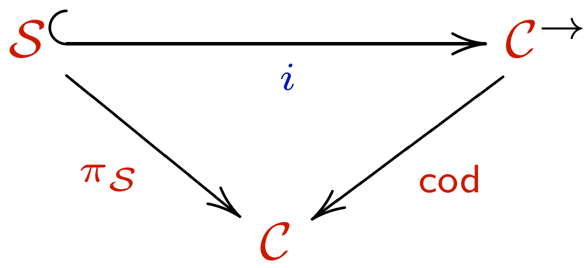
$$\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{C}$$

satisfying the following properties:

- the category $\mathcal{C}$ has finite limits
  ($\mathcal{C}$ is meant to be the category of collections)

- the subobject doctrine $\mathbf{Sub}_{\mathcal{C}}$ associated to $\mathcal{C}$ is a first order Lawvere hyperdoctrine
  (represents the logic over collections)

- the fibration $\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{C}$ is a FULL sub-fibration of the codomain fibration on $\mathcal{C}$
  ($\pi_{\mathcal{S}}$ represent family of sets indexed over classes)



i.e. i is an inclusion functor preserving cartesian morphisms and making the diagram commute.

- for each object A of $\mathcal{C}$ the fibre $\mathcal{S}_A$ of $\pi_{\mathcal{S}}$ over A is a locally cartesian closed pretopos;

- for any morphism $f\colon A \to B$ the substitution functor

$$f^*\colon \mathcal{S}_B \to \mathcal{S}_A$$

preserves the LCC pretopos structure;

- for each object A of $\mathcal{C}$ the embedding of each fibre $\mathcal{S}_A$ into $\mathcal{C}/A$ preserves the LCC pretopos structure;

- there is a $\mathcal{C}$-object Ω
  classifying the subobjects of $\mathcal{C}$ which are in $\mathcal{S}$:

  $$\mathsf{Sub}_{\mathcal{S}} \simeq \mathcal{C}(-,\Omega)$$

  where $\mathsf{Sub}_{\mathcal{S}}(A)$ is the full subcategory of $\mathsf{Sub}_{\mathcal{C}}(A)$ of those subobjects which are represented by objects in $\mathcal{S}$;

- for every $\mathcal{C}$-object A,
  for every object $\alpha: X \to A$ in $\mathcal{S}$,
  there is an *exponential object* $(\pi_\Omega)^\alpha$ in $\mathcal{C}/A$
  where $\pi_\Omega: A \times \Omega \to A$ is the first projection, i.e. there is a natural isomorphism

  $$\mathcal{C}/A(- \times \alpha\,,\ \pi_\Omega) \simeq \mathcal{C}/A(-\,,\ (\pi_\Omega)^\alpha)$$

  as functors on $\mathcal{C}/A$.

Our examples of predicatively generalized elementary toposes

In our next examples of predicatively generalized elementary toposes

$\pi_{\mathcal{S}}: \mathcal{S} \rightarrow \mathcal{C}$

we just specify $\mathcal{C}$ and $\mathcal{S}$

since $\pi_{\mathcal{S}}$ must be the *restriction* of the codomain fibration

Elementary toposes are examples of our structures

An **elementary topos** $\mathcal{T}$ is a **predicatively generalized elementary topos**

with collections=sets:

$\mathcal{S} = \mathcal{C}^{\rightarrow} = \mathcal{T}^{\rightarrow}$

$\pi_{\mathcal{S}} = cod_{\mathcal{T}}$

i.e.

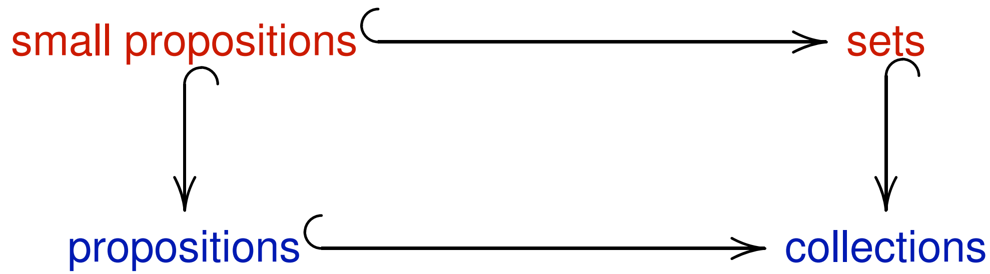
$\pi_{\mathcal{T}} : \mathcal{T}^{\rightarrow} \rightarrow \mathcal{T}$

The internal language of pred. gen. elem. toposes

The **generic internal language** $\mathcal{T}_{p-top}$ of predicatively generalized elementary toposes

is an extension of first order extensional Martin-Löf's type theory

with four entities, called sorts



collections	closed under type constructors of a lex category closed under power collections of a set
propositions	as at most singleton collections closed under first order connectives
sets	closed under type constructors of an LCC pretopos
small propositions (derived sort)	defined as at most singleton sets

Theorem: TT_{ptop} extended with the following resizing rule

$$\frac{A \text{ proposition}}{A \text{ small proposition}}$$

becomes equivalent to the generic internal language TT_{topoi} of elementary toposes.

A syntactic example with Aczel's CZF

we build a predicatively generalized elementary topos

based on entities formalized in

Aczel's Constructive Zermelo Fraenkel axiomatic set theory **CZF**

$\mathcal{C} =$	syntactic category of **CZF**-definable classes and functional relations
Objects of $\mathcal{S}$ =	projections of disjoint sums of indexed families of sets $B(x) set \ (x \in A)$ over a class A denoted with $\pi_1 : \Sigma_{x \in A} B(x) \ \rightarrow \ A$

Key points:

Ω=	Power class $\{x \mid x \subseteq \mathbf{1}\}$
	of subsets of the chosen terminal object 1
Ω^A=	Power class of subsets of the set A

A syntactic example with *Voevodsky*'s Univalent Foundations

we consider

HoTT$_0$ = Homotopy Type Theory

i.e. Martin-Löf's intensional type theory + *Voevodsky*'s Univalence Axiom

with *one universe* U$_0$

closed under codes for higher effective quotients A/R

relative to a set A in U$_0$

and to an equivalence relation R represented by a proposition in U$_0$

both according to the notion of set/proposition by *Voevodsky* and the associated syntactic category

Set(**HoTT**$_0$)	
objects	*Voevodsky*'s sets of **HoTT**$_0$
morphisms	typed terms (under renaming of variables)
morphism equality	propositional equality of terms

and we build a predicatively generalized elementary topos with

$\mathcal{C}$	= Set(**HoTT**$_0$)
Objects of $\mathcal{S}$	projections of families of sets in the first universe U_0
	i.e. of dependent sets $B(x)\ set\ [x \in A]$
	such that $B(x) \in \mathsf{U}_0\ [x \in A]$

Key points:

thanks to the Univalence axiom:
extensionality of functions holds in $\mathcal{C}$ + $\Omega = \Sigma_{p \in \mathsf{U}_0}\ \mathsf{isprop}(P)$ is the type of propositions in the first universe (which is a set) with propositional equality given by **equiprovability**
higher inductive quotients of sets

see [the HoTT Book]

and for the LCC pretopos structure

[E. Rijke, B. Spitters'14] “Sets in homotopy type theory”, MSCS

Boolean predicative toposes

Prop.

Given a predicatively generalized elementary topos

$\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{C}$

if $Sub_{\mathcal{C}}$ is boolean then $\mathcal{S}/1$ is a boolean elementary topos.

To build more examples of predicative toposes

we need of a predicative analogue

of Johnstone-Hyland-Pitts tripos-to-topos construction

we adopt

the exact completion of a weakly finite limit category

in [Carboni'95]

called **exact on lex completion**

but viewed as an elementary quotient completion

to better compute with it.

a comparison between the two completions

the tripos-to-topos one and the exact on lex one

is given in

[M.E.M. , F.Pasquali, G. Rosolini'17] ”Triposes, exact completions, and Hilbert’s -operator”

in Tbilisi Mathematical Journal

Generalization of exact on lex completion

the **Elementary Quotient Completion**

of a Lawvere's **Elementary doctrine**

introduced in

[M.E.M. and G. Rosolini'13] "Quotient completion for the foundation of constructive mathematics", Logica Universalis

[M..E.M. and G.-Rosolini'13] ""Elementary quotient completion", Theory and Applications of Categories

Basic notion: Elementary Doctrine

An **elementary doctrine** is an indexed inf-semilattice (hence supporting logic of *conjunction*)

$$P\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$$

with $\mathscr{C}$ with binary products
and supporting *logical equality*, i.e. for each A in $Ob\mathscr{C}$, there is an object δ_A in $ObP(A \times A)$ s.t.

(i) for α in $P(A)$

$$\exists_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge \delta_A$$

is a left adjoint to the substitution along a diagonal

$P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle} \colon P(A \times A) \to P(A)$

(ii) *the left adjoint lifts to a fibre with an extra context:* for α in $P(X \times A)$

$$\exists_{\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle}(\alpha) := P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\alpha) \wedge_{A \times A} P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta_A)$$

is a left adjoint to $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle} \colon P(X \times A \times A) \to P(X \times A)$.

Elementary Quotient Completion $\mathscr{Q}_P$

the Elementary Quotient Completion $\mathscr{Q}_P$ of an elementary doctrine

$$P \colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSL}$$

has:

objects (A, ρ) with ρ a P-equivalence relation on A

(written in the logic of P)

an arrow $\lfloor f \rceil : (A, \rho) \to (B, \sigma)$ equivalence class of arrows $f \colon A \to B$ in $\mathscr{C}$

$$\rho \leq_{A \times A} P_{f \times f}(\sigma)$$

with

$$f =_{\mathscr{Q}_P} g \qquad \text{iff} \qquad \rho \leq P_{f \times g}(\sigma)$$

Fibres of $\overline{P}$

for (A, ρ) object of $\mathscr{Q}_P$

$$\overline{P}(A, \rho) := \mathscr{D}\textit{es}_\rho$$

with $\alpha \in \mathscr{D}\textit{es}_\rho$ descend data,

i.e.

$$P_{\mathrm{pr}_1}(\alpha) \wedge \rho \leq P_{\mathrm{pr}_2}(\alpha)$$

where $\mathrm{pr}_1, \mathrm{pr}_2 \colon A \times A \to A$ are the projections.

Original motivation for the elementary quotient completion

the **Elementary quotient completion**

gives an algebraic axiomatization of the quotient/setoid model

used to interpret the extensional level of **MF**

into its intensional one

in [M'09]

in terms of its universal properties.

On NON-exactness of Elementary Quotient Completions

for suitable P

$\mathcal{Q}_P$ is not EXACT

whilst REGULAR!!

(every equivalence relation in $\mathcal{Q}_P$ has a stable coequalizer)

examples of NON exact elementary quotient completions

The elementary quotient completion $\mathcal{Q}_P$ is NOT exact

when

P= logic of Coquand's Calculus of Constructions

or

P= logic of the intensional level of the Minimalist Foundation

When is the Elementary Quotient Completion $\mathcal{Q}_P$ exact?

Theorem:

The Elementary Quotient Completion $\mathcal{Q}_P$ of an elementary- variational existential- doctrine P

is exact

iff

$$P \simeq \Psi_{\mathscr{C}}$$

i.e. the starting elementary doctrine P is equivalent to the doctrine of weak subobjects on $\mathscr{C}$

$$\Psi_{\mathscr{C}} : \mathscr{C}^{\text{op}} \longrightarrow \text{InfSL}$$

"posetal collapse" of the pullback pseudo-functor

i.e. its fibre over a $\mathscr{C}$-object A is

$\Psi_{\mathscr{C}}(A)$= the posetal collapse of $\mathscr{C}/A$

from [M.E.M., F.Pasquali, G. Rosolini'17] "Triposes, exact completions, and Hilbert's -operator" Tbilisi Mathematical Journal

[M.E.M., G. Rosolini'16] "Relating quotient completions via categorical logic"

A syntactic example with Martin-Löf's type theory

we consider

MLTT$_0$ = Martin-Löf's type theory with one universe

and the associated syntactic category

C(**MLTT**$_0$)	
objects	closed types of **MLTT**$_0$
morphisms	typed terms (under renaming of variables)
morphism equality	propositional equality of terms

and we build a predicatively generalized elementary topos with

$\mathcal{C}$	= setoid model à la Bishop over **MLTT**$_0$ = exact on lex completion over C(**MLTT**$_0$)
Objects of $\mathcal{S}$	objects of $\mathcal{C}^{\rightarrow}$ isomorphic in the fibre over their codomain denoted with $A_{=}$, to projections of $A_{=}$-indexed family of setoids à la Bishop-Richman indicated with the notation $B(x)_{=}(x)\ [\,x \in A_{=}\,]$ where both support $B(x)$ and equivalence relation $=_{B(x)}$ are in the first universe U$_0$

Dependent setoid $B_=(x)\ [\, x \in A_= \,]$ by Bishop-Richman

a functor

$$B(-) \quad : \quad Gpd(A_=) \longrightarrow \mathcal{C}$$

$Gpd(A_=)$ = posetal groupoid category associated to the setoid $A_=$

(also as internal action on the internal groupoid $Gpd(A_=)$ in $\mathcal{C}$)

Dependent setoid $B_=(x)\ [\, x \in A_= \,]$ by Bishop-Richman

$B(x)\ set\ [\, x \in A \,]$ called "dependent support"

+ an equivalence relation $=_{B(x)}$

+ *substitution (iso)morphisms*

$\sigma_{x_1}^{x_2} : B(x_1)_= \longrightarrow B(x_2)_=$ for $x_1 \in A,\ x_2 \in A,$ proof of $x_1 =_A x_2$

- sending $=_{B(x_1)}$ to $=_{B(x_2)}$
- *NON depending* on the proof of $x_1 =_A x_2$
- $\sigma_x^x : B(x)_= \to B(x)_=$ is the identity
- the $\sigma_{x_1}^{x_2}$'s are closed under composition:

$\sigma_{x_2}{}^{x_3} \cdot \sigma_{x_1}{}^{x_2} =_{\mathrm{Q(mTT)}} \sigma_{x_1}{}^{x_3}$

for equivalent x_1, x_2, x_3

A syntactic example with Feferman's Theory of NON-iterative fixpoints

In **Feferman's Theory of NON-iterative fixpoints** $\widehat{ID_1}$

we use *formulas defining fixpoints* of so called admissible formulas

to define

a **universe of $I\hat{D}_1$-sets** $U_0^{I\hat{D}_1}$
a notion of **$I\hat{D}_1$-small proposition** as a $I\hat{D}_1$-set which is at most singleton

exactly as that used in

[I. Ishihara, M.E.M., S. Maschio, T.Streicher'18]

"Consistency of the Minimalist Foundation with Church's thesis and Axiom of Choice", *AML*.

A predicative version of Hyland's Effective Topos

Let $\mathbf{Rec}^{I\hat{D}_1}$ be the following category

objects	definable classes in $I\hat{D}_1$ (i.e. subsets of natural numbers defined by formulas $\phi(x)$ up to renaming of variables)
morphisms	recursive functions in $\widehat{ID_1}$ denoted by numerals
morphism equality	extensional equality

we define a predicatively generalize elementary topos

meant to be a predicative version of *Hyland*'s Effective Topos

with:

$\mathcal{C}_{pEff}$	= the exact on lex completion of $\mathbf{Rec}^{I\hat{D}_1}$ viewed as elementary quotient completion
Objects of $\mathcal{S}_{pEff}$	= objects of $\mathcal{C}^{\rightarrow}_{pEff}$ isomorphic in the fibre over their codomain $A_=$ to projections of $A_=$-indexed families of objects in $\mathcal{C}_{pEff}$ whose support is in $\mathsf{U_0}^{I\hat{D}_1}$ and whose equivalence relation is a small proposition

the interpretation of the *logical connectives and quantifiers*

in the hyperdoctrine structure of the subobject functor

is equivalent to Kleene realizability interpretation of intuitionistic logic.

in [M.E. Maietti and S. Maschio'18] "A predicative variant of Hyland's Effective Topos" on ArXiv the category of collections $\mathcal{C}_{pEff}$ of our predicatively generalized elementary topos

can be embedded in *Hyland*'s Effective Topos Eff

thanks to the fact that Eff is an exact on lex completion on partioned assemblies

by embedding the category $\mathbf{Rec}^{I\hat{D}_1}$ of recursive functions in $\widehat{ID_1}$

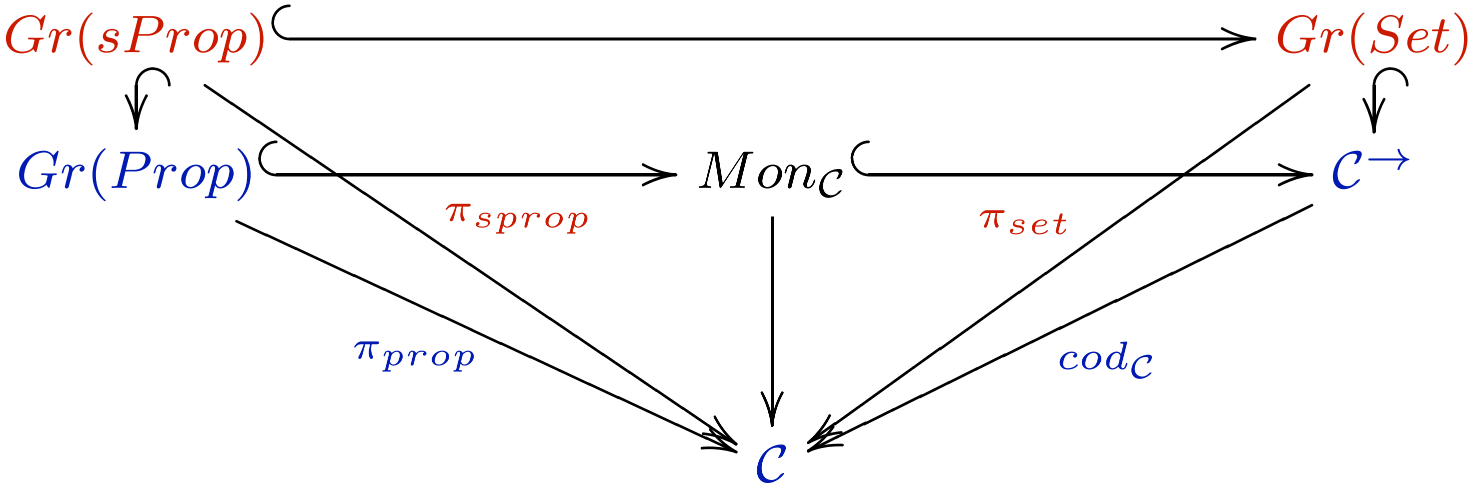
in the corresponding category of subsets of natural numbers and recursive functions in Eff.

Predicative generalization of arithmetic quasi-topos

An **arithmetic quasi-topos** is a quasi-topos with a natural numbers object and closed under *stable effective quotients* of *strong equivalence relations*.

A **predicatively generalized arithmetic quasi-topos** is a tuple of full sub-fibered categories of the codomain fibration of a lex category $\mathcal{C}$ (meant to be collections)

$$(\mathcal{C} \, , \; \pi_{set} \, , \pi_{prop} \, , \pi_{sprop} \;)$$



where all the inclusion are cartesian FULL embeddings

modelling respectively

the category of collections,

the fibration of collection indexed families of propositions

that of collection indexed families of sets

that of collection indexed families of small propositions

all formalized in the extensional level of the **Minimalist Foundation**.

Internal language of predicatively generalized arithmetic quasi-toposes

The extensional level of the **Minimalist Foundation** in [M'09]

(which is an extensional dependent type theory)

provides the *internal language*

of our notion of predicatively generalized arithmetic quasi-topos

Future work

- Build examples predicative toposes and predicative quasi-topos
  for consistency/conservativity results about the Minimalist Foundation by applying topos techniques

- Develop a notion of *internal sheaf* in a predicatively generalized elementary topos
  by using *Formal Topology*
  and relate it to the work by **Moerdijk-Palmgren-van den Berg**