Models of Univalence in Toposes

Andrew Pitts

UNIVERSITY OF CAMBRIDGE

Computer Science & Technology

Joint work with Ian Orton, Dan Licata & Bas Spitters

Plan

- Universes in toposes
- Univalence: extensionality for universes
- Identity types from intervals in toposes
- Univalent universes from tiny intervals
- Conclusion

Universes in a topos $\mathcal{E}$

Types whose elements denote types are crucial for the expressive power of Martin-Löf Type Theory (MLTT).

Modelling MLTT in toposes leads one to consider universe objects

$$\begin{array}{c}\tilde{\mathcal{U}} \\ \downarrow \mathbf{p} \\ \mathcal{U}\end{array}$$

a family of $\mathcal{E}$-objects, indexed by the generalized elements of $\mathcal{U}$

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a generalized element u at stage $\Gamma \in \mathcal{E}$

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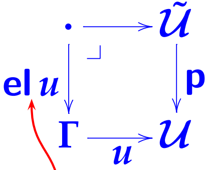

a generalized element u at stage $\Gamma \in \mathcal{E}$
gives an object $\mathbf{el}\,u$ in $\mathcal{E}/\Gamma$ by pulling back $\mathbf{p}$ along u:
u is a **code** for the object $\mathbf{el}\,u$

Universes in a topos $\mathcal{E}$

Bénabou; Hofmann-Streicher: can transfer a notion of size in $\mathbf{Set}$ (Grothendieck universe) to give a corresponding universe in any topos bounded over $\mathbf{Set}$ (Grothendieck topos).

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Assumption: will only consider toposes $\mathcal{E}$ equipped with a sequence of universes $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots$ that are

- nested: $\mathcal{S}_n$ has a code in $\mathcal{S}_{n+1}$
- internal full subtoposes, closed under taking dependent products, etc. [details omitted]

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However, there's more to universes than just questions of size...

Extensionality for universes

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What about universes? In that case the data are codes for types.

$$\begin{array}{ccc} \cdot & \longrightarrow & \tilde{\mathcal{U}} \\ {\scriptstyle \mathbf{el}\, u}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathbf{p}} \\ \Gamma & \xrightarrow[u]{} & \mathcal{U} \end{array}$$

How do we explain when two such codes are equal?

$$\forall u, v \in_\Gamma \mathcal{U}, \qquad u = v \;\Leftrightarrow\; \quad ?$$

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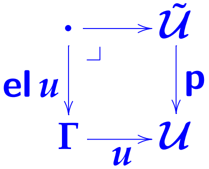
$$\forall u, v \in_\Gamma \mathcal{U}, \qquad u = v \Leftrightarrow \mathbf{el}\,u = \mathbf{el}\,v$$

Can hold for universes of sets ($\mathcal{E} = \mathbf{Set}$), but is mathematically uninteresting – in general, we should only care about objects up to **isomorphism** ($\cong$), not equality ($=$).

Extensionality for universes



$$\forall u, v \in_\Gamma \mathcal{U}, \qquad u = v \Leftrightarrow \mathbf{el}\, u \cong \mathbf{el}\, v$$

A too strong, “skeletal” property – hard to achieve and anyway it doesn’t take account of the different ways that u and v can be isomorphic.

Extensionality for universes

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$$\forall u, v \in_\Gamma \mathcal{U}, \qquad \mathbf{ua} : (u \sim v) \simeq (\mathbf{el}\, u \simeq \mathbf{el}\, v)$$

Key idea (Voevodsky's univalence axiom): replace predicate "$u = v$" by **identity types** $u \sim v$ (generalized elements of which are like proofs of equality of u and v) and compare it with an object of **equivalences** ($\simeq$) between $\mathbf{el}\, u$ and $\mathbf{el}\, v$
($\simeq$ is itself defined in terms of $\sim$, rather than $=$).

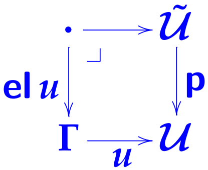
While discussing **ua**, allow me to blur the distinction between

syntax	&	**semantics**
Γ **context**		$\Gamma \in \mathcal{E}$
$\Gamma \vdash A$ **type**		$\begin{array}{c} \bullet \\ \downarrow{\scriptstyle A} \\ \Gamma \end{array} \in \mathcal{E}/\Gamma$
$\Gamma \vdash a : A$		$\begin{array}{ccc} \Gamma & \xrightarrow{a} & \bullet \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow{\scriptstyle A} \\ & & \Gamma \end{array}$
$\prod_{x:A} B(x)$		local exponentials

etc. . .

and to ride rough shod over questions of how to deal with "strictness" (semantics of substitution).

Extensionality for universes



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Identity types

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$$\frac{p : a \sim_A a'}{\textbf{contr} : (a, \textbf{refl}) \sim_{\sum_{x:A}(a \sim_A x)} (a', p)}$$

$$\frac{x : A \vdash B(x) \ \textbf{type} \qquad p : a \sim_A a' \qquad b : B(a)}{p * a : B(a')}$$

(Adapting a trick due to **Lumsdaine**, given the above, we can always redefine the transport operation $*$ to ensure there is a proof of $\textbf{refl} * a \sim_A a$ and hence, by **Coquand-Danielsson**, of identity induction.)

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$\mathcal{E}$'s extensional equality $(=)$ gives identity types

$$a \sim_A a' \ \triangleq \ \{x : 1 \mid a = a' : A\}$$

that are too "thin" to satisfy **ua** (unless $\mathcal{E}$ is degenerate). A more intensional notion of equality is needed for **ua**.

Homotopical view of identity types

space	A **type**
point	$a : A$
path	$p : a \sim_A a'$
homotopy	$\pi : p \sim_{(a \sim_A a')} p'$
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Voevodsky *et al* [arXiv:1211.2851v4]: the topos of simplicial sets (in classical ZFC) provides a concrete instance of this analogy and has univalent universes w.r.t. usual Kan simplicial path types.

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Cohen-Coquand-Huber-Mörtberg (CCHM) [arXiv:1611.02108]: a presheaf topos of "cubical sets" (in IZF, say) that provides a constructive model of univalence.

There are now several variations on cubical sets.

Orton-AMP [arXiv:1712.04864], **Licata-Orton-AMP-Spitters** [arXiv:1801.07664]: an elementary axiomatic treatment of what makes these models of univalence tick.

Given $0, 1 : \mathbf{1} \rightrightarrows \mathbb{I}$ in some topos $\mathcal{E}$,
what is required of the "interval" object $\mathbb{I}$ to make
$$a \sim_A a' \triangleq \{p : A^{\mathbb{I}} \mid p\,0 = a \wedge p\,1 = a'\}$$
an identity type for which there is a univalent universe?

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provided $\mathbb{I}$ has a connection $\_ \sqcap \_ : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$, a weak form of binary minimum satisfying some simple axioms.
- To get transport operations $\_ * \_ : (p : a \sim_A a') \to B(a) \to B(a')$ for families $x : A \vdash B(x)$, we equip them with extra structure – CCHM fibration structure...

CCHM fibration structure

...is a form of (uniform) Kan-filling operation with respect to "cofibrations":

Assumption: $\mathcal{E}$ comes equipped with a class of monos, called cofibrations, generated under pullback from a generic one $\mathbf{1} \rightarrowtail \mathbf{Cof}$ (where $\mathbf{Cof} \rightarrowtail \mathbf{\Omega}$) satisfying... [axioms t.b.a.]

CCHM fibration structure

Given a family of types $A : \Gamma \to \mathcal{S}$,
a CCHM fibration structure $\alpha : \mathbf{Fib}\, A$ maps

path in Γ	$p : \Gamma^{\mathbb{I}}$
cofibrant partial path over p	$f : \prod_{i:\mathbb{I}}(\varphi \to A(p\, i))$ with $\varphi : \mathbf{Cof}$
extension of f at $\mathtt{0}$	$a_0 : A(p\, \mathtt{0})$ with $f\, \mathtt{0} \nearrow a_0$

to

extension of f at $\mathtt{1}$	$a_1 : A(p\, \mathtt{1})$ with $f\, \mathtt{1} \nearrow a_1$

where extension relation for $\varphi : \mathbf{Cof}$, $f : \varphi \to \Gamma$ and $x : \Gamma$ is

$$f \nearrow x \triangleq \prod_{u:\varphi}(f\, u = x)$$ "f agrees with x where defined"

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(In particular, so long as $\mathsf{false} : \mathsf{Cof}$, then each $\alpha : \mathsf{Fib}\, A$ induces a transport function $\_ * \_ : \prod_{p\,:\,\Gamma^{\mathbb{I}}}(A\,0 \to A\,1)$, namely $p * a = \pi_1(\alpha\, p\, \mathsf{false}\,!\,(a, !))$.)

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Some simple closure properties of $\mathbf{Cof}$ enable one to prove that the existence of fibration structure is preserved under forming $\boldsymbol{\Sigma}$-types, $\boldsymbol{\Pi}$-types, path types,…

What about universes of fibrations?

Tiny interval

$\mathcal{F}(\Gamma) \triangleq \sum_{A:\Gamma\to\mathcal{S}} \mathsf{Fib}\, A$

set of CCHM fibrations over an object $\Gamma \in \mathcal{E}$

Theorem. If interval $\mathbb{I} \in \mathcal{E}$ is tiny, i.e. $(\_)^{\mathbb{I}}$ has a right adjoint (denoted $\surd : \mathcal{E} \to \mathcal{E}$), then $\mathcal{F}(\_) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ is representable and we get a universe of CCHM fibrations.

Theorem generalizes unpublished work of **Coquand & Sattler** for the case $\mathcal{E}$ is a presheaf topos.

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When $\mathcal{E} = \mathbf{Set}^{\Box^{\mathbf{op}}}$, the topos of cubical sets,

the category $\Box$ has finite products

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Hence the path functor $(\_)^{\mathbb{I}} : \mathbf{Set}^{\Box^{\mathrm{op}}} \to \mathbf{Set}^{\Box^{\mathrm{op}}}$ is $(\_ \times I)^*$

and so $(\_)^{\mathbb{I}}$ not only has a left adjoint $(\_ \times \mathbb{I})$, but also a right adjoint.

Classifying fibration

The Theorem says that one can use tinyness of $\mathbb{I}$ to construct a

classifying fibration $(\mathcal{U} \xrightarrow{\pi_1} \mathcal{S}, \nu : \mathsf{Fib}\,\pi_1) \in \mathcal{F}(\mathcal{U})$ satisfying:

for all $(\Gamma \xrightarrow{A} \mathcal{S}, \alpha : \mathsf{Fib}\,A) \in \mathcal{F}(\Gamma)$

there is a unique $\ulcorner A, \alpha \urcorner : \Gamma \to \mathcal{U}$ such that

$(A, \alpha) = \mathcal{F}(\ulcorner A, \alpha \urcorner)(\pi_1, \nu)$

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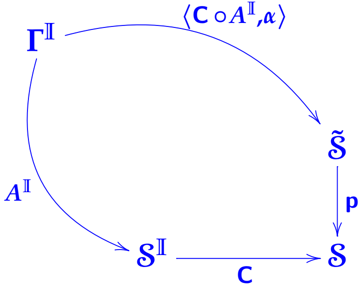

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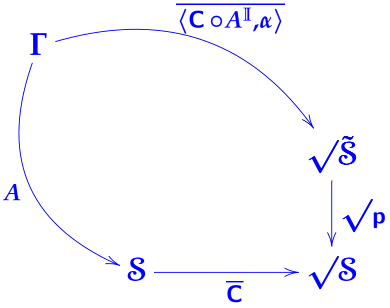

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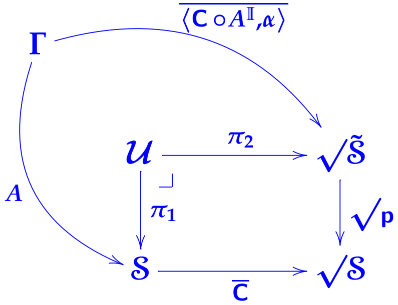

$\mathcal{U} \triangleq$ pullback of $\overline{\mathsf{C}}$ and $\sqrt{\mathsf{p}}$

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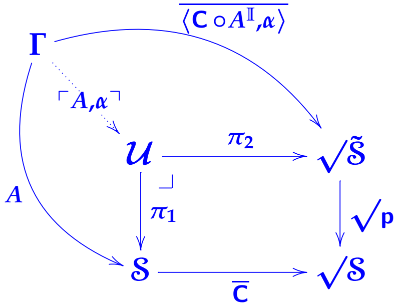

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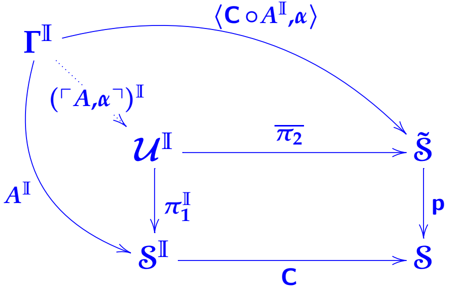

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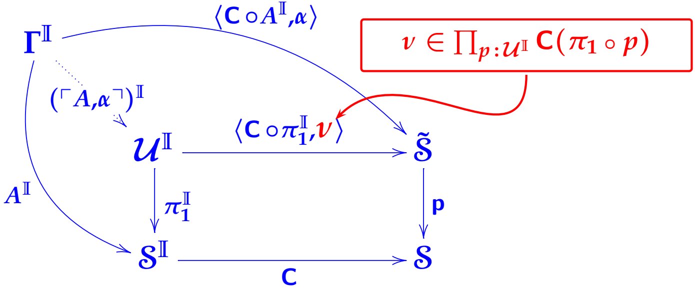

uniqueness of $\ulcorner A, \alpha \urcorner$ follows from
uniqueness property of the pullback square

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and from that a universe, by pulling back:

$$\begin{array}{ccc} \tilde{\mathcal{U}} & \longrightarrow & \tilde{\mathcal{S}} \\ \big\downarrow & \lrcorner & \big\downarrow \mathbf{p} \\ \mathcal{U} & \xrightarrow[\pi_1]{} & \mathcal{S} \end{array}$$

Theorem. Universe $\tilde{\mathcal{U}} \to \mathcal{U}$ is closed under Π-types, $\sim$-types and inductive types (e.g. Σ) if $\mathbf{Cof}$ satisfies

$$\mathbf{false} \in \mathbf{Cof}$$

$$\forall i : \mathbb{I}, \varphi : \mathbf{Cof},\ (\varphi \vee i = 0 \in \mathbf{Cof})\ \wedge\ (\varphi \vee i = 1 \in \mathbf{Cof})$$

What about univalence of $\tilde{\mathcal{U}} \to \mathcal{U}$?

Homotopical view of identity types

space	A **type**
point	$a : A$
path	$p : a \sim_A a'$
homotopy	$\pi : p \sim_{(a \sim_A a')} p'$
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One of Voevodsky's key insights: the role of contractibility

$$\textbf{isContr}\, A \;\triangleq\; \textstyle\sum_{a:A} \prod_{a':A} (a \sim_A a')$$

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$$\mathsf{isEquiv}\, f \triangleq \textstyle\prod_{b:B} \mathsf{isContr}(\sum_{a:A} (f\, a \sim_B b))$$

$$A \simeq B \triangleq \textstyle\sum_{f:B^A} \mathsf{isEquiv}\, f$$

$$\mathsf{UA}(\mathcal{U}, \mathsf{el}) \triangleq \textstyle\prod_{u,v:\mathcal{U}} \mathsf{isEquiv}(u \sim_{\mathcal{U}} v \to \mathsf{el}\, u \simeq \mathsf{el}\, v)$$

Univalence

Theorem. For any topos $\mathcal{E}$ with $\mathbb{I}$ & **Cof** satisfying assumptions so far, there is a term of type $\mathbf{UA}(\mathcal{U}, \mathbf{el})$ if

> **Cof** is closed under $(\forall i : \mathbb{I})\_$ and
> satisfies the isomorphism extension axiom...

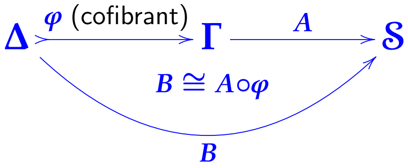
In this case $\mathcal{U} \to \mathbf{1}$ is a fibration and $(\mathcal{U}, \mathbf{el})$ is univalent.

Proof is non-trivial.

Isomorphism extension axiom

$$\mathbf{iea} : \prod_{A:\mathcal{S}} \mathbf{Ext}\left(\sum_{B:\mathcal{S}} (A \cong B)\right)$$

(where $\mathbf{Ext}\,\Gamma \triangleq \prod_{\varphi\,:\,\mathbf{Cof}} \prod_{f\,:\,\varphi \to \Gamma} \sum_{x:\Gamma} (f \nearrow x)$)



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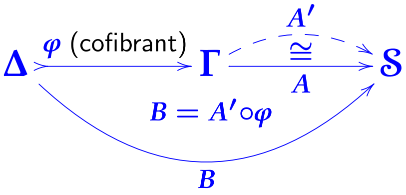

Isomorphism extension axiom

$$\mathbf{iea} : \prod_{A:\mathcal{S}} \mathsf{Ext}\left(\sum_{B:\mathcal{S}} (A \cong B)\right)$$

(where $\mathsf{Ext}\,\Gamma \triangleq \prod_{\varphi:\mathsf{Cof}} \prod_{f:\varphi\to\Gamma} \sum_{x:\Gamma} (f \nearrow x)$)

Theorem. In a presheaf topos $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, Cof satisfies $\mathbf{iea}$ if for each $X \in \mathbf{C}$ and $S \in \mathsf{Cof}(X) \subseteq \Omega(X)$, the sieve S is a decidable subset of $\mathbf{C}/X$.
(So with classical meta-theory, it always holds for presheaf toposes.)

For the CCHM topos of cubical sets $\mathbf{Set}^{\square^{\mathrm{op}}}$, the authors make a particular choice for Cof that does have this decidability property (and which is closed under $(\_)^{\mathbb{I}}$).

Summary of assumptions

- Elementary topos $\mathcal{E}$ with universes
  $\mathbf{\Omega} : \mathcal{S}_0 : \mathcal{S}_1 : \mathcal{S}_2 : \cdots$
- "Interval" object $\mathbb{I}$ (in $\mathcal{S}_0$) which has distinct end-points & connection operation (& for convenience, a reversal operation)
  and which is tiny.
- Universe of "cofibrant" propositions $\mathbf{Cof} \rightarrowtail \mathbf{\Omega}$
  containing **false**, $i = 0$, $i = 1$,
  closed under $\_ \vee \_$ and $\forall(i : \mathbb{I})\_$
  and satisfying the isomorphism extension axiom.

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets, because the path functor is fibered over $\mathcal{E}$ and we can use internal language to describe many of the constructions on the way to a univalent universe. . .

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets. because the path functor is fibered over $\mathcal{E}$ and we can use internal language to describe many of the constructions on the way to a univalent universe...

  ...but not all of them: tinyness does not internalize! (so neither does our universe construction)

  Licata-Orton-AMP-Spitters use type theory enriched with a modality for global properties ("crisp type theory") in order to express the whole construction with a type-theoretic language.

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- The axiomatic approach helps one see the wood from the trees in existing models and to find new ones
  (cf. recent work of Uemura using the effective topos)

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- The axiomatic approach helps one see the wood from the trees in existing models and to find new ones
- Nevertheless, some of the theorems on the way to univalence/fibrancy are delicate and hard work!

  We find the use of an interactive theorem proving system (Agda-flat) invaluable for developing and checking the proof – e.g. see [doi.org/10.17863/CAM.21675]

Conclusion

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  Are there simpler models of univalence? (must be non-truncated to qualify for our attention)

  E.g. can one avoid Kan-filling in favour of a (weak) notion of path composition?

  Why only presheaf toposes?

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- The axiomatic approach helps one see the wood from the trees in existing models and to find new ones
- Nevertheless, some of the theorems on the way to univalence/fibrancy are delicate and hard work!
- Further reading:

I. Orton and A. M. Pitts, *Axioms for Modelling Cubical Type Theory in a Topos* [arXiv:1712.04864]

D. R. Licata, I. Orton, A. M. Pitts and B. Spitters, *Internal Universes in Models of Homotopy Type Theory* [arXiv: 1801.07664]

Thank you for your attention!