

# Nonconnective structures

Toposes in Como

Università dell'Insubria

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# Plan of the talk

- Motivations: HKR theorems in analytic geometry
- Background on derived analytic geometry
- Nonconnective structures

# Motivations: HKR theorems in analytic geometry

Let  $k$  be a field and let  $A$  be a  $k$ -algebra. The  $i$ th Hochschild homology group is

$$\mathrm{HH}_i(A) := \mathrm{Tor}_i^{A \otimes_k A}(A, A) \simeq H^i(A \otimes_{A \otimes_k A}^L A).$$

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## Theorem (HKR)

*Let  $k$  be a field of characteristic zero and let  $A$  be a smooth  $k$ -algebra. Then there is a multiplicative isomorphism*

$$\mathrm{HH}_*(A) \simeq \Lambda^* \Omega_{A/k}^1.$$

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**Consequence:** the correct guess becomes

$$A \otimes_{A \otimes_k A}^L A \simeq \mathrm{Sym}_A(\mathbb{L}_{A/k}[1]).$$

We would like the above equivalence to be multiplicative.



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sending  $A$  to  $A \otimes_{A \otimes_k A}^L A$  and

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**Problem:** the right adjoints do not exist!

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### Proposition

*The functors  $\mathrm{HH, DR}: \mathrm{cdga}_k \rightarrow \mathrm{cdga}_k^{\leq 0}$  admit right adjoints.*

# The right adjoint in the de Hochschild case

Since we work in the commutative setting, we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{cdga}_k}(\mathrm{HH}(A), B) &\simeq \mathrm{Map}_{\mathrm{cdga}_k}(A \otimes_{A \otimes_k A}^L A, B) \\ &\simeq \mathrm{Map}(A, B) \times_{\mathrm{Map}(A, B) \times \mathrm{Map}(A, B)} \mathrm{Map}(A, B) \\ &\simeq \mathrm{Map}_{\mathrm{cdga}_k}(A, B \times_{B \times B} B)\end{aligned}$$



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So the functor

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**Remark:**  $B \times_{B \times B} B \simeq B \otimes_k (k \times_{k \times k} k) \simeq B \oplus B[-1]$ .

# The right adjoint in the de Rham case

Fix  $f: A \rightarrow B$ . We have a fiber sequence

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{cdga}_k/A}(\mathrm{Sym}_A(\mathbb{L}_A[1]), B) & \longrightarrow & \mathrm{Map}_{\mathrm{cdga}_k}(\mathrm{Sym}_A(\mathbb{L}_A[1]), B) \\ \downarrow & & \downarrow \\ \{*\} & \xrightarrow{f} & \mathrm{Map}_{\mathrm{cdga}_k}(A, B) \end{array}$$

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**Conclusion:** the right adjoint to DR is given by the  $\infty$ -functor

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**Remark:**  $B \oplus B[-1] \simeq B \otimes_k (k \oplus k[-1])$ .

**Observe:** there is a multiplicative zig-zag

$$\begin{array}{ccc} & \text{Sym}_k(k[-1]) & \\ & \swarrow \quad \searrow & \\ k \oplus k[-1] & & k \times_{k \times k} k. \end{array}$$

When  $\text{char}(k) = 0$ , these are equivalences.

# The goal

The HKR theorem can be reformulated as follows:

## Theorem

Let  $X$  be a scheme over  $k$ . There is an equivalence of **derived** schemes

$$X \times_{X \times X}^d X \simeq T^*X[-1],$$

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This same statement makes sense in both complex analytic and nonarchimedean analytic (Berkovich) geometry.

**Our goal:** prove the analytic version.

This is an ongoing project with J. António and F. Petit.

# Background on derived analytic geometry

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So one might expect to obtain the category of derived analytic spaces as a full subcategory of the  $\infty$ -category of simplicially ringed spaces (or, better,  $\infty$ -topoi). For example the next definition sounds reasonable:

## Definition (Wrong)

A derived analytic space is a pair  $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ , where:

- 1  $\mathcal{X}$  is the topos of sheaves on either a  $\mathbb{C}$ -analytic space or on the quasi-étale site of a Berkovich space;
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There are many (equivalent) ways of explaining the problem with the above definition:

- 1 the associated deformation theory is not what we expect;
- 2 Lurie's representability theorem fails;
- 3 the category of derived analytic spaces (in the above sense) is not closed under pushouts along closed immersions.

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## Definition

Let  $(X, \mathcal{O}_X)$  be an analytic space and let  $F$  be a (discrete) coherent sheaf on  $X$ . An analytic derivation on  $X$  with values in  $F$  is a derivation  $\mathcal{O}_X \rightarrow F$  which furthermore satisfies

$$d(f(x)) = f'(x)dx$$

for every  $x \in \mathcal{O}_X(U)$  and every analytic function  $f: \mathbf{A}^1 \rightarrow \mathbf{A}^1$ .



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To do this in a sufficiently canonical way, we need to endow the structure sheaf of our derived analytic spaces with an *extra analytic structure*.

# Analytic structure sheaves

Essentially two different approaches to formalize the idea:

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- Interpret  $\mathcal{O}_{\mathcal{X}}$  as a sheaf of simplicial algebras equipped with an axiomatic holomorphic functional calculus (Lurie, P., P. - Yu).

Here it is a recap on functional calculus. Let  $A$  is a commutative Banach  $\mathbb{C}$ -algebra. Any element  $a \in A$  determines a solid arrow

$$\begin{array}{ccc} \mathbb{C}[z] & \xrightarrow{a} & A \\ \downarrow & \nearrow & \\ \mathcal{H}(U) & & \end{array}$$

If  $U \subset \mathbb{C}$  is an open subset and  $\mathcal{H}(U)$  denotes the ring of holomorphic functions on  $U$ , we have a restriction  $\mathbb{C}[z] \rightarrow \mathcal{H}(U)$ .

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**Answer:** Let  $\sigma(a) \subset \mathbb{C}$  denote the spectrum of  $a \in A$ : it is the image of the Gelfand transform of  $a$ , seen as a function  $\mathcal{M}(A) \rightarrow \mathbb{C}$ . Then the lifting problem has solution if and only if  $\sigma(a) \subset U$ .

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**Axiomatic holomorphic functional calculus:** (Rough) An algebra  $A$  together with subsets  $A(U) \subset A^n$  for every  $U \subset \mathbb{C}^n$  and (composable) holomorphic operations  $f_A: A(U) \rightarrow A$  for every  $f: U \rightarrow \mathbb{C}$ .

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Let  $\mathcal{T}_{\text{an}}(\mathbb{C})$  be the category of open subsets of  $\mathbb{C}^n$ , with holomorphic functions between them.



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**Reformulation:** an axiomatic holomorphic functional calculus on a commutative  $\mathbb{C}$ -algebra  $A$  is a product-preserving functor

$$\mathcal{O}: \mathcal{T}_{\text{an}}(\mathbb{C}) \rightarrow \text{Set}$$

such that  $\mathcal{O}(\mathbb{C}) = A$ .

The  $k$ -analytic analogue of  $\mathcal{T}_{\text{an}}(\mathbb{C})$  is the following:

### Definition

$\mathcal{T}_{\text{an}}(k)$  is the full subcategory of  $\text{An}_k$  spanned by separated, quasi-smooth strictly  $k$ -analytic spaces which are paracompact.

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Let  $\mathcal{T}_{\text{an}}$  be either  $\mathcal{T}_{\text{an}}(\mathbb{C})$  or  $\mathcal{T}_{\text{an}}(k)$ . Let  $\mathcal{X}$  be an  $\infty$ -topos. A sheaf of analytic rings on  $\mathcal{X}$  is a functor  $\mathcal{O}: \mathcal{T}_{\text{an}} \rightarrow \mathcal{X}$  which preserves:

- 1 products;
- 2 pullbacks along open immersions (in the  $\mathbb{C}$ -analytic case) or quasi-étale maps (in the  $k$ -analytic case).

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To any analytically ringed topos  $(\mathcal{X}, \mathcal{O})$ , we can always associate a simplicially ringed topos  $(\mathcal{X}, \mathcal{O}^{\text{alg}})$ .

## Definition (Rough)

A *derived analytic space* is an analytically ringed space  $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$

- 1  $(\mathcal{X}, \pi_0(\mathcal{O}_{\mathcal{X}}^{\text{alg}}))$  is a classical  $\mathbb{C}$ -analytic space (or a Berkovich space);
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## Theorem (Lurie, P. - Yu)

- 1 *There exists an  $\infty$ -category  $\text{dAn}_{\mathbb{C}}$  (resp.  $\text{dAn}_k$ ) of derived  $\mathbb{C}$ -analytic (resp.  $k$ -analytic) spaces.*
- 2  *$\text{dAn}_{\mathbb{C}}$  and  $\text{dAn}_k$  admit fiber products.*
- 3 *There are fully faithful embeddings  $\text{An}_{\mathbb{C}} \rightarrow \text{dAn}_{\mathbb{C}}$  and  $\text{An}_k \rightarrow \text{dAn}_k$ .*
- 4 *If  $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$  is a derived analytic space, then so is  $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$  for every  $n \geq 0$ .*

## Theorem (P. - Yu)

Let  $F: \mathrm{dStn}_{\mathbb{C}}^{\mathrm{op}} \rightarrow \mathcal{S}$  be a sheaf for  $\tau_{\mathrm{an}}$ . The following conditions are equivalent:

- 1  $F$  is a derived analytic Artin stack;

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  - (iv)  $F$  admits a global analytic cotangent complex.

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**Idea:** enlarge  $\mathcal{T}_{\text{an}}(k)$  by adding the higher classifying stacks  $B^n(\mathbf{G}_a)$ : let  $\mathcal{T}_{\text{an}}^{\text{nc}}(k)$  be the smallest full subcategory of analytic Artin stacks closed under products and containing both  $\mathcal{T}_{\text{an}}(k)$  and the collection  $\{B^n(\mathbf{G}_a)\}$ .



## Definition

Let  $\mathcal{X}$  be an  $\infty$ -topos. A  $\mathcal{T}_{\text{an}}^{\text{nc}}(k)$ -nonconnective structure is a functor

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- 1 it commutes with products;
- 2 the restriction to  $\mathcal{T}_{\text{an}}(k)$  is a  $\mathcal{T}_{\text{an}}(k)$ -structure;
- 3 for every  $n \geq 0$  the diagram

$$\begin{array}{ccc} \mathcal{O}(B^n(\mathbf{G}_a)) & \longrightarrow & \mathcal{O}(\text{Spec}(k)) \\ \downarrow & & \downarrow \\ \mathcal{O}(\text{Spec}(k)) & \longrightarrow & \mathcal{O}(B^{n+1}(\mathbf{G}_a)) \end{array}$$

is a pullback.

## Remark

The collection  $\{\mathcal{O}(B^n(\mathbf{G}_a))\}_{n \geq 0}$  forms an  $\Omega$ -spectrum in  $\mathcal{X}$ . This gives an *underlying spectrum* functor

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## Theorem (António, Petit, P.)

*The connective cover functor*

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*induced by precomposition with  $\mathcal{T}_{\mathrm{an}}(k) \hookrightarrow \mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)$  has a fully faithful left adjoint*

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Thanks for the attention!