Grothendieck categories as a bilocalization of linear sites

Toposes in Como

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Motivation

Noncommutative Algebraic Geometry

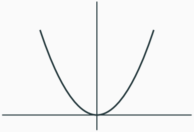
Algebra Geometry

Noncommutative Algebraic Geometry

Algebra

$A = k[x, y]/\langle y - x^2 \rangle$

Geometry



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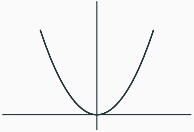

Category Theory

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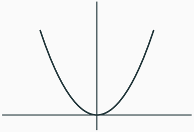
$A = k[x,y]/\langle y - x^2 \rangle$		$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y-x^2 \rangle}\right)$

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Noncommutative Algebraic Geometry

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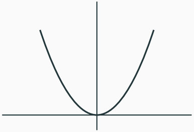
$A = k[x,y]/\langle y - x^2 \rangle$		$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y-x^2\rangle}\right)$
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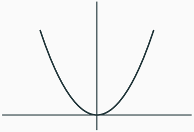
$A = k[x, y]/\langle y - x^2 \rangle$		$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y-x^2 \rangle}\right)$
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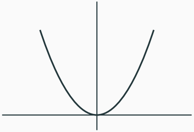
$A = k[x,y]/\langle y - x^2 \rangle$		$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y-x^2\rangle}\right)$
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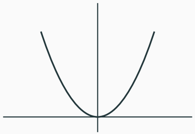
$A = k[x,y]/\langle y - x^2 \rangle$		$\mathrm{Qch}(\mathrm{Sp}(A)) = \mathrm{Mod}\left(\frac{k[x,y]}{\langle y - x^2\rangle}\right)$
Noncommutative multiplication	?????	Categorical information is still available!
A NC ring	$\to$	Mod(A) - affine NC space
A NC graded ring + ...	$\to$	Qgr(A) - projective NC space (Serre's thm)

Which categories?

Grothendieck categories will be our models for NC spaces

Grothendieck categories

k – commutative ring

Definition

A *k-linear Grothendieck category* is a cocomplete k-linear abelian category with a generator and exact filtered colimits.

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Motivation

Theorem (Gabber)

Let X be a scheme. $\mathrm{Qch}(X)$ *is a Grothendieck abelian category.*

Theorem (Gabriel-Rosenberg)

We can recover the geometry of a scheme X from $\mathrm{Qch}(X)$.

Linear sites and Grothendieck categories: the objects

Linear sites

$\mathfrak{a}$ – small k-linear category $\equiv$ enriched over $\mathrm{Mod}(k)$

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Definition

A *k-linear Grothendieck topology* $\mathcal{T}$ on $\mathfrak{a}$ is a $\mathrm{Mod}(k)$-enriched version of the classical notion of Grothendieck topology, i.e. for every $A \in \mathfrak{a}$ the covering sieves are submodules $R \subseteq \mathfrak{a}(-, A)$ fulfilling the usual axioms of a Grothendieck topology in the enriched setup.

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A *k-linear site* is a pair $(\mathfrak{a}, \mathcal{T})$ as above.

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Definition

A *k-linear site* is a pair $(\mathfrak{a}, \mathcal{T})$ as above.

This is one instance of the enriched sheaf theory introduced by Borceux and Quinteiro in [BQ96].

This particular example has been analysed later on by Lowen in [Low16] with deformation theory purposes.

Linear sheaves and presheaves

$(\mathfrak{a}, \mathcal{T})$ – k-linear site

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Definition

- *k-linear presheaves*: $\mathsf{Mod}(\mathfrak{a}) := \mathsf{Fun}_k(\mathfrak{a}^{\mathrm{op}}, \mathsf{Mod}(k))$
- *k-linear sheaves*: $\mathsf{Sh}(\mathfrak{a}, \mathcal{T}) \subseteq \mathsf{Mod}(\mathfrak{a})$ the full subcategory of k-linear presheaves F such that the restriction

$$Mod(\mathfrak{a})(\mathfrak{a}(-, A), F) \xrightarrow{\cong} \mathsf{Mod}(\mathfrak{a})(R, F),$$

for all $A \in \mathfrak{a}$ and all $R \in \mathcal{T}(A)$.

Linear sheaves and presheaves

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Proposition (Borceux-Quinteiro)

The inclusion $\mathrm{Sh}(\mathfrak{a}, \mathcal{T}) \subseteq \mathrm{Mod}(\mathfrak{a})$ *is a localization functor. Its k-linear exact left adjoint* $\# : \mathrm{Mod}(\mathfrak{a}) \longrightarrow \mathrm{Sh}(\mathfrak{a}, \mathcal{T})$ *is called* sheafification.

The Gabriel-Popescu theorem

Theorem (Gabriel-Popescu, generalization by Lowen)
k-Linear Grothendieck categories are the categories of sheaves over k-linear sites (k-linear Giraud theorem).

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Remark

For each Grothendieck category $\mathcal{C}$, there exist multiple choices of linear sites $(\mathfrak{a}, \mathcal{T})$ such that $\mathcal{C} \cong \mathrm{Sh}(\mathfrak{a}, \mathcal{T})$.

Grothendieck categories and linear sites: the (1-)categories

The (1)-categories of Grothendieck categories

For **geometric** interests:

- The (1-)category Grt with:
  - $\mathrm{Obj}(\mathsf{Grt}) = \{k\text{-linear Grothendieck categories}\}$
  - $\mathsf{Grt}(\mathcal{A}, \mathcal{B}) = \{\text{colimit preserving } k\text{-linear functors } \mathcal{A} \to \mathcal{B}\}$

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For **classical topos theory** interests:

- The (1-)category Topos_k with:
  - $\mathrm{Obj}(\mathrm{Topos}_k) = \{k\text{-linear Grothendieck categories}\}$
  - $\mathrm{Topos}_k(\mathcal{A}, \mathcal{B}) = \{\text{geometric } k\text{-linear functors } F^* : \mathcal{B} \rightleftarrows \mathcal{A} : F_*\}$

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For **geometric** interests:

- The (1-)category $\mathsf{Site}_{k,\mathrm{cont}}$ with:
  - $\mathrm{Obj}(\mathsf{Site}_{k,\mathrm{cont}}) = \{k\text{-linear sites}\}$
  - $\mathsf{Site}_{k,\mathrm{cont}}((\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}), (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})) =$
    $\{\text{continuous } k\text{-linear functors } f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})\} =$
    $\{k\text{-linear } f : \mathfrak{a} \to \mathfrak{b} \mid f^* : \mathsf{Mod}(\mathfrak{b}) \to \mathsf{Mod}(\mathfrak{a}) : M \mapsto M \circ f$
    restricts to a map $f_s : \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \to \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})\}$

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Recall:

Proposition (SGA 4)
If $f : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \longrightarrow (\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ is a continuous morphism, there exists a functor $f^s : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \longrightarrow \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ such that $f^s \dashv f_s$, and in particular, it is colimit preserving.

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For **classical topos theory** interests:

- The (1-)category Site_k with:
  - $\mathrm{Obj}(\mathsf{Site}_k) = \{k\text{-linear sites}\}$
  - $\mathsf{Site}_k((\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}), (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})) =$
    $\{k\text{-linear morphisms of sites } f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})\} =$
    $\{\text{continuous } k\text{-linear functors } f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \text{ such that}$
    $f^s : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \text{ is exact}\}$

From linear sites to Grothendieck categories

We can define (pseudo)functors:

- $\phi : \mathsf{Site}_{k,\mathrm{cont}} \longrightarrow \mathsf{Grt}$ given by:

$$\phi(\mathfrak{a}, \mathcal{T}_\mathfrak{a}) := \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$$

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- $\psi : \mathsf{Site}_{k} \longrightarrow \mathsf{Topos}_k^{\mathrm{op}}$ given by:

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The localization intuition

Theorem (SGA 4, Lowen, ...)

Let $f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \longrightarrow (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ *be a continuous* k*-linear morphism such that* $f^{-1}\mathcal{T}_{\mathfrak{b}} = \mathcal{T}_{\mathfrak{a}}$ *and satisfying:*

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Then $f_s : \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \to \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ *is an equivalence, and hence so is* f^s*. A continuous* f *with those properties is called an LC morphism.*

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Remark

Observe that every LC morphism is in particular a k-linear morphism of sites, hence we have:

$$\mathrm{LC}((\mathfrak{a}, \mathcal{T}_\mathfrak{a}), (\mathfrak{b}, \mathcal{T}_\mathfrak{b})) \subseteq \mathrm{Site}_k((\mathfrak{a}, \mathcal{T}_\mathfrak{a}), (\mathfrak{b}, \mathcal{T}_\mathfrak{b})) \subseteq \mathrm{Site}_{k,\mathrm{cont}}((\mathfrak{a}, \mathcal{T}_\mathfrak{a}), (\mathfrak{b}, \mathcal{T}_\mathfrak{b}))$$

The roof theorem

Theorem (Stacks Project + RG)

Given $F : \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \to \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ *colimit preserving, there exists a subcanonical site* $(\mathfrak{c}, \mathcal{T}_\mathfrak{c})$ *and continuous morphisms* f, u *as in*

$$\begin{array}{ccc} & (\mathfrak{c}, \mathcal{T}_\mathfrak{c}) & \\ \overset{f}{\nearrow} & & \overset{u}{\nwarrow} \\ (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) & & (\mathfrak{b}, \mathcal{T}_\mathfrak{b}), \end{array}$$

with u *an LC morphism, such that* $F = u_s \circ f^s$.

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GABRIEL-ZISMAN LOCALIZATION: Morphisms in Grt are obtained inverting LC morphisms in $\mathrm{Site}_{k,\mathrm{cont}}$.

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GABRIEL-ZISMAN LOCALIZATION: Morphisms in $\mathrm{Topos}_k^{\mathrm{op}}$ are obtained inverting LC morphisms in Site_k.

Localization in the bicategorical setup

The 2-categories

For **geometric** interests, we consider:

- The 2-category Grt as before, with 2-morphisms given by the k-linear natural transformations $A : F \Rightarrow G$ between colimit preserving functors
- The 2-category $\mathrm{Site}_{k,\mathrm{cont}}$ as before, with 2-morphisms given by the k-linear natural trasnformations $\alpha : f \Rightarrow g$ between continous functors

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For **classical topos theory** interests, we consider:

- The 2-category Topos_k as before, with 2-morphisms given by the k-linear natural transformations $A : F_* \Rightarrow G_*$ between the right adjoints of the geometric morphisms
- The 2-category Site_k as before, with 2-morphisms given by the k-linear natural trasnformations $\alpha : f \Rightarrow g$ between morphisms of sites

Pronk's bicategories of fractions

Pronk introduces in [Pro96] a suitable generalization to the bicategorical setting of the 1-categorical notion of a class of (1-)morphisms admitting a left/right calculus of fractions and defines bicategories of fractions in this setup.

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Definition (Pronk)

Let $\mathcal{C}$ be a bicategory and W a class of 1-morphisms admitting a left calculus of fractions. A *bilocalization of* $\mathcal{C}$ *along* W is a pair $(\mathcal{C}[\mathsf{W}^{-1}], \Lambda)$ of a bicategory and a pseudofunctor such that:

1. $\Lambda : \mathcal{C} \to \mathcal{C}[\mathsf{W}^{-1}]$ sends elements in W to equivalences;
2. Composition with Λ gives an equivalence of bicategories

$$\mathrm{Hom}(\mathcal{C}[\mathsf{W}^{-1}], \mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathsf{W}}(\mathcal{C}, \mathcal{D})$$

for each bicategory $\mathcal{D}$.

Main result

Proposition

LC *admits a left calculus of fractions in* $\mathrm{Site}_{k,\mathrm{cont}}$.

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Theorem ([Ram18])

There exists a pseudofunctor

$$\Phi : \mathsf{Site}_{k,\mathrm{cont}} \to \mathsf{Grt}$$

which sends LC morphisms to equivalences in Grt*, such that the pseudofunctor*

$$\tilde{\Phi} : \mathsf{Site}_{k,\mathrm{cont}}[\mathrm{LC}^{-1}] \to \mathsf{Grt}$$

induced by Φ *via the universal property of the bilocalization is an equivalence of bicategories.*

Main result

Proposition

LC *admits a left calculus of fractions in* Site_k.

Theorem ([Ram18])

There exists a pseudofunctor

$$\Psi : \mathsf{Site}_k \to \mathsf{Topos}_k^{\mathrm{coop}}$$

which sends LC morphisms to equivalences in $\mathsf{Topos}_k^{\mathrm{coop}}$, *such that the pseudofunctor*

$$\tilde{\Psi} : \mathsf{Site}_k[\mathrm{LC}^{-1}] \to \mathsf{Topos}_k^{\mathrm{coop}}$$

induced by Ψ *via the universal property of the bilocalization is an equivalence of bicategories.*

Sketch of the proof

1. **Extend** $\phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 1**-categories to a pseudofunctor** $\Phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 2**-categories**:

For $f, g : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ continuous functors,

$$\Phi(\alpha : f \Rightarrow g) := \alpha^s : f^s \Rightarrow g^s$$

defined by adjunction from $\alpha_s : g_s \Rightarrow f_s$, where

$$(\alpha_s)_F(A) := F(\alpha_A) : g_s(F)(A) = F(g(A)) \to F(f(A)) = f_s(F)(A),$$

for all $F \in \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$, all $A \in \mathfrak{a}$.

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2. **Observe** Φ **sends LC morphisms to equivalences**.

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3. **One proves that** Φ **satisfies Tommasini's criterion.**

Sketch of the proof

Tommasini's criterion [Tom]: Provides necessary and sufficient conditions for $A : \mathcal{C} \to \mathcal{D}$ sending a class W of 1-morphisms with a left calculus of fractions in $\mathcal{C}$ to equivalences in $\mathcal{D}$, so that its induced pseudofunctor $\tilde{A} : \mathcal{C}[\mathsf{W}^{-1}] \to \mathcal{D}$ is an equivalence of bicategories.

Sketch of the proof

1. **Extend** $\phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 1**-categories to a pseudofunctor** $\Phi : \mathrm{Site}_{k,\mathrm{cont}} \to \mathrm{Grt}$ **between the** 2**-categories**:

For $f, g : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ continuous functors,

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2. **Observe** Φ **sends LC morphisms to equivalences**.
3. **One proves that** Φ **satisfies Tommasini's criterion.**

Sketch of the proof

1. Extend $\psi : \mathrm{Site}_k \to \mathrm{Topos}_k^{\mathrm{op}}$ between the 1-categories to a pseudofunctor $\Psi : \mathrm{Site}_k \to \mathrm{Topos}_k^{\mathrm{coop}}$ between the 2-categories:

For $f, g : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \to (\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ morphisms of sites,

$$\Psi(\alpha : f \Rightarrow g) := \alpha_s : g_s \Rightarrow f_s$$

defined as

$$(\alpha_s)_F(A) := F(\alpha_A) : g_s(F)(A) = F(g(A)) \to F(f(A)) = f_s(F)(A),$$

for all $F \in \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$, all $A \in \mathfrak{a}$.

2. Observe Ψ sends LC morphisms to equivalences.
3. One proves that Ψ satisfies Tommasini's criterion.

Remarks

- The analogous results can be obtained analogously in the set-theoretical setup.

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- We can safely define constructions in the 2-category of Grothendieck categories at the level of the sites as long as they behave well with respect to LC morphisms

One instance: Monoidal structure

Our original motivation: Study of a monoidal structure in Grt

- In [LRS17] we have defined a tensor product of Grothendieck categories based on a tensor product of linear sites, which defines a monoidal structure in $\mathsf{Site}_{k,\mathrm{cont}}$.

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- In addition, it is also shown in [LRS17] that LC morphisms are closed with respect to this tensor product of sites.
- A bicategorical version of monoidal localization à la Day [Day73], developed in unpublished work by Pronk, would immediately provide us with a monoidal structure in Grt, where the tensor product "is given" by the one defined in [LRS17].

Thank you for your attention

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