Synthetic topology in Homotopy Type Theory for probabilistic programming

Bas Spitters
Two papers

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Synthetic topology in Homotopy Type Theory for probabilistic programming

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An Application of Computable Distributions to the Semantics of Probabilistic Programs
Application of topos theory

A research program on:
higher topos theory and homotopy type theory
Monadic programming with effects

Moggi’s computational λ-calculus

Kleisli’s category of a monad:

- $\text{Obj}(\mathcal{C}_T) = \text{Obj}(\mathcal{C})$;
- $\mathcal{C}_T(A, B) = \mathcal{C}(A, T(B))$.

Used for:

Undefinedness: $X + \bot$

State: $(X \times S)^S$

Non-determinism: $\mathcal{P}(X)$

Discrete probabilities: $\text{convex}(X)$
Desiderata for a framework for the semantics of probabilistic programming languages?

Such a framework should support:

- continuous and discrete types, and a rich variety of functions between them,
- random choice $+_r$,
- choosing from a rich variety of discrete and continuous distributions,
- conditioning,
- a rich type system including sum, product, function, and probability types,
- recursive definitions of values, and
- recursive definitions of types.
Probability theory

- **Classical probability**: measures on \(\sigma\)-algebras of sets
  \(\sigma\)-algebra: collection closed under countable \(\cup, \cap\)
  measure: \(\sigma\)-additive map to \(\mathbb{R}\).

- **Giry monad**:
  \(X \mapsto \text{Meas}(X)\) is a monad
  on measurable spaces, on subcategories of topological spaces or domains.
  valuations restrict measures to opens.

Topology in a topos ...
Synthetic topology

Scott: Synthetic domain theory
Domains as sets in a topos (Hyland, Rosolini, ...)
By adding axioms to the topos we make a DSL for domains.

Synthetic topology
(Brouwer, ..., Escardo, Taylor, Vickers, Bauer, ...)
Every object carries a topology, all maps are continuous
Idea: Sierpinski space $\Sigma = (\emptyset)$ classifies opens:

$$O(X) \cong X \to \Sigma$$

Convenient category of/type theory for ‘topological’ spaces.

Synthetic (real) computability
semi-decidable truth values $\Sigma$ classify semi-decidable subsets.

Common generalization based on abstract properties for $\Sigma \subset \Omega$:
Dominance axiom: maps classified by $\Sigma$ compose.
More axioms for synthetic topology

Let $N^\bullet$ be the type of increasing binary sequences ‘the one-point compactification of $N$’.  

**WSO** (‘Weakly Sequentially Open’):
The intrinsic topology, $N^\bullet \to \Sigma$, coincides with the metric topology $d(n, m) = 2^{-\min(n,m)}$:
If $f : N^\bullet \to \Sigma$ and $f(\infty) = 1$, then there exists $n$ s.t. $f(n) = 1$.

**WSO** contradicts classical logic, but holds in our models.

A stronger principle:
**Fan**: $2^\mathbb{N}$ is metrizable and compact

Lešnik developed analysis synthetically from these principles.
**Countable choice** is often not needed.

Fix such a topos where every object comes with a topology.
Realizability topos

PCA: Partial Combinatorial Algebra
Model of the untyped $\lambda$-calculus
Examples:
- $K_1$ Turing machines
- $K_2$ Turing machines with infinite I/O-tapes.

Sets with a computability structures
Can be made into a topos (Hyland, Pitts)

Embedding of realizability models into sheaf models
(Awodey/Bauer)
Big Topos

Topological site:
A category of topological spaces closed under open inclusions
Covering by jointly epi families
Big topos: sheaves over such a site

Model for intuitionism: all maps are continuous
Nice category vs nice objects
Valuations and Lower integrals

Lower Reals:
\( r : \mathbb{R}_l := \mathbb{Q} \to \mathbb{S} \)
\( \forall p, r(p) \iff \exists q, (p < q) \land r(q). \)
\( \rightsquigarrow \) lower semi-continuous topology.

Dedekind Reals:
\( \mathbb{R}_D := (\mathbb{Q} \to \mathbb{S}) \times (\mathbb{Q} \to \mathbb{S}) \)
\( \text{lower real} \times \text{upper real} \)

Valuations:
Valuations on \( A : \text{Set} : \)
\( Val(A) = (A \to \mathbb{S}) \to \mathbb{R}_l^+ \)
  - \( \mu(\emptyset) = 0 \)
  - Modularity
  - Monotonicity
  - Continuity

Integrals:
Positive integrals:
\( Int^+(A) = (A \to \mathbb{R}_D^+) \to \mathbb{R}_D^+ \)
  - \( \int (\lambda x.0) = 0 \)
  - Additivity
  - Monotonicity
  - Probability: \( \int \lambda .1 = 1 \)

Riesz theorem: homeomorphism between integrals and valuations.
Constructive proof (Coquand/S): \( A \) regular compact locale.
Valuations and Lower integrals

Lower Reals:
\[ r : \mathbb{R}_l := \mathbb{Q} \to \mathbb{S} \]
\[ \forall p, r(p) \iff \exists q, (p < q) \land r(q). \]
\[ \leadsto \text{lower semi-continuous topology.} \]

Dedekind Reals:
\[ \mathbb{R}_D := (\mathbb{Q} \to \mathbb{S}) \times (\mathbb{Q} \to \mathbb{S}) \]
\[ \text{lower real} \times \text{upper real} \]

Valuations:
Valuations on \( A : \text{Set} \):
\[ \text{Val}(A) = (A \to \mathbb{S}) \to \mathbb{R}_l^+ \]
\[ \text{Modularity} \]
\[ \text{Monotonicity} \]
\[ \text{Continuity} \]

Lower integrals:
Positive integrals:
\[ \text{Int}^+(A) = (A \to \mathbb{R}_l^+) \to \mathbb{R}_l^+ \]
\[ \int (\lambda x.0) = 0 \]
\[ \text{Additivity} \]
\[ \text{Probability} \]
\[ \text{Monotonicity} \]

Riesz theorem: homeomorphism between integrals and valuations.
Monadic semantics

Giry monad: (space) ↦ (space of its valuations):

- functor $\mathcal{M} : Space \rightarrow Space$.
- unit operator $\eta_x = \delta_x$ (Dirac)
- bind operator $(I \gg= M)(f) = \int_I \lambda x. (Mx)f$.

$(\gg=) : \mathcal{MA} \rightarrow (A \rightarrow \mathcal{MB}) \rightarrow \mathcal{MB}$. 
To interpret the full computational $\lambda$-calculus we need $T$-exponents $(A \to TB)$.

The standard Giry monads do not support this.

Set is cartesian closed, so we obtain a higher order language. Moreover, the Kleisli category is $\omega$-cpo enriched (we use subprobability valuations), so we can interpret fixed points.

Rich semantics for a programming language, as requested by Plotkin.
Huang developed an efficient compiled higher order probabilistic programming language: augur/v2

Semantics in topological domains
(domains with computability structure)

**Theorem**

*The interpretation of the monadic calculus in the realizability topos gives the same interpretation.*
Type theory

Formalizing this construction in homotopy type theory.

- Correctness
- Programming language with an expressive type system
Discrete probabilities: ALEA library

**ALEA library** (Audebaud, Paulin-Mohring) basis for CertiCrypt

- Discrete measure theory in Coq;
- Monadic approach (Giry, Jones/Plotkin, ...):
  
  \[
  \text{CPS:} \quad (A \to [0, 1]) \to [0, 1]
  \]

  \[
  \text{submonad: monotonicity, summability, linearity.}
  \]

  Coq cannot prove that this is a monad (no funext).

**Example:** flip coin: \( Mbool \)

\[
\lambda (f : bool \to [0, 1]).(0.5 \times f(\text{true}) + 0.5 \times f(\text{false}))
\]

First question: Can we avoid ‘setoid hell’?
Univalent homotopy type theory

Coq lacks quotient types and functional extensionality. ALEA uses setoids, \((T, \equiv)\). (‘exact completion’)

Univalent homotopy type theory: an internal type theory for a generalization of setoids, groupoids, ...
We use Coq’s HoTT library.
(CPP: Bauer, Gross, Lumsdaine, Shulman, Sozeau, Spitters)
Toposes and types

How to formalize toposes in type theory? Use HoTT as a language for higher toposes. Rijke/S: hSets in HoTT form a (predicative) topos: large power objects.

Conjecture (Shulman,...): Both Grothendieck toposes and realizability can be lifted to HoTT. Partial results:
- Simplicial sheaves (Cisinski/Shulman)
- Cubical stacks (Coquand)
- Cubical assemblies (Uemura, CMU)
- Cubical model in NuPrl (Bickford, Coquand, Mörtberg)
- Internal models (last talk)

Here: we show how this is useful. Our second use of HoTT: Predicative constructive maths without countable choice.
Implementation in HoTT

Our basis: Cauchy reals in HoTT as HIIT (book, Gilbert)

- HoTTClasses: like MathClasses but for HoTT
- Experimental Induction-Recursion branch by Sozeau

Partiality (Altenkirch, Danielson): Construction in HoTT:
free $\omega$-cpo completion as a higher inductive inductive type:

$$A_\bot : hSet \quad \bot : A_\bot \quad \eta : A \to A_\bot \quad \subseteq_{A_\bot} : A_\bot \to A_\bot \to Type$$

$$\bigcup : \prod_{f : \mathbb{N} \to A_\bot} (\prod_{n : \mathbb{N}} f(n) \subseteq_{A_\bot} f(n + 1)) \to A_\bot \quad \subseteq \text{must satisfy the expected relations.}$$

$\subseteq$:=Partial(1) as $\Sigma$. 
Higher order probabilistic computation (Related work)

Compare: Top is not Cartesian closed.
1. Define a convenient super category. E.g. quasi-topological spaces: concrete sheaves over compact Hausdorff spaces. This is a quasi-topos which models synthetic topology. Even: big topos
2. Add probabilities inside this setting.

Staton, Yang, Heunen, Kammar, Wood model for higher order probabilistic programming has the same ingredients (but in opposite direction):
1. Standard Giry model for probabilistic computation
2. Obtain higher order by (a tailored) Yoneda
Conclusions

- Probabilistic computation with continuous data types
- Formalization in HoTT
- Experiment with synthetic topology in HoTT
- Extension of the Giry monad from locales to synthetic topology
- Model for higher order probabilistic computation: Augur/v2